Vieta's Formulas Handout Answers and Solutions

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1 Answers

1. 16

2. 78

3. 7

4. -2

5. 35

6. 1

7. 330

8. 256

9. $3+\sqrt{7}$

10. 0

11. 6

12. $-\frac{3}{11}$

13. 3

14. 753

15. -24

2 Solutions

1. The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

Solution: Let the roots be b and c. Then we have that $a = b + c$ and $2a = bc$. It follows that $bc = 2b + 2c$. We can rewrite this equation as $(b-2)(c-2) = 4$. Clearly, the only integer solutions to this equation are $(b, c) = (6, 3), (4, 4), (3, 6), (1, -2), (0, 0)$, and $(-2, 1)$. The corresponding values of a are $9, 8, -1$, and 0. It follows that our answer is $9 + 8 - 1 = \boxed{16}$ as desired.

2. The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of a?

Solution: Let the roots be p, q, and r. We know that $pqr = 2010$, and we wish to minimize $p + q + r$. $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. We need one of p, q, and r to be divisible by 67, and because 67 is such a large prime, it makes sense to let one of p, q, and r to be equal to 67. WLOG, let $p = 67$. It follows that $qr = 30$. By $AM - GM$, the sum $q + r$ will be minimized when q and r are as close to $\sqrt{30}$ as possible. It follows that the sum will be minimized when (q, r) is a permutation of $(5, 6)$. Therefore, the minimum value of a is $67 + 5 + 6 = \boxed{78}$.

3. Let P be a cubic monic polynomial with roots a, b, and c. If $P(1) = 91$ and $P(-1) = -121$, compute the maximum possible value of

$$\frac{ab + bc + ca}{abc + a + b + c}.$$

Solution: We can write $P(x)$ as $P(x) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$. We know that $P(1) = (1+ab+ac+bc)-(a+b+c+abc) = 91$ and that $-1-(a+b+c)-(ab+ac+bc)-abc = -121$. Letting $1 + ab + ac + bc = x$ and $a + b + c + abc = y$, it follows that $x - y = 91$ and $x + y = 121$, and therefore $x = \frac{91+121}{2} = 106$ and $y = 106 - 91 = 15$. Therefore, the maximum value of $\frac{ab+bc+ca}{abc+a+b+c}$ is the maximum value of $\frac{x-1}{y}$ which is $\frac{105}{15} = \boxed{7}$.

4. Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

Solution: We will begin by finding the polynomial P which has roots $a + 1$, $b + 1$, and $c + 1$. We know that $a + 1 + b + 1 + c + 1 = a + b + c + 3 = 3$. We know that $(a + 1)(b + 1) + (a + 1)(c + 1) + (b + 1)(c + 1) = ab + ac + bc + 2(a + b + c) + 3 = -1 + 3 = 2$. Finally, we know that $(a + 1)(b + 1)(c + 1) = abc + ab + ac + bc + a + b + c + 1 = -1 - 1 + 1 = -1$. Therefore, a polynomial P with roots of $a+1$, $b+1$, and $c+1$ is $P(x) = x^3 - 3x + 2x^2 + 1$. We wish to determine $\frac{(a+1)(b+1)+(a+1)(c+1)+(b+1)(c+1)}{(a+1)(b+1)(c+1)} = \frac{2}{-1} = \boxed{-2}$ as desired.

5. Find the sum of all the real values of x satisfying $(x + \frac{1}{x} - 17)^2 = x + \frac{1}{x} + 17$.

Solution: Let $x + \frac{1}{x} - 17 = a$. Then we have that $a^2 = a + 34$. Solving, we get $a = \frac{1\pm\sqrt{137}}{2}$. This means that if we let $x + \frac{1}{x} = b$, then we have that $b = \frac{35\pm\sqrt{127}}{2}$. First we will look at the case when $x + \frac{1}{x} = \frac{35+\sqrt{127}}{2}$. Rearranging, we get $x^2 - (\frac{35+\sqrt{127}}{2})x + 1 = 0$. By Vieta's Formulas, the sum of the values of x which satisfy this is $\frac{35+\sqrt{127}}{2}$. When $x + \frac{1}{x} = \frac{35-\sqrt{127}}{2}$, we can similarly find that the sum of the values of x which satisfy this is $\frac{35-\sqrt{127}}{2}$. Because these two quadratics do not share any roots, the sum of the values of x which satisfy the original equation is $\frac{70}{2} = \boxed{35}$.

6. The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. Compute the absolute value of h.

Solution: Let the roots be a and b. Then we know $a + b = -2h$ and that $ab = -3$. We also know that $a^2 + b^2 = (a + b)^2 - 2ab = 4h^2 + 6 = 10$. It follows that $h^2 = 1$, and therefore, the absolute value of h is $\boxed{1}$.

7. Find the sum of the real roots of the polynomial

$$\prod_{k=1}^{100}(x^2 - 11x + k) = (x^2 - 11x + 1)(x^2 - 11x + 2)...(x^2 - 11x + 100).$$

Solution: We can remember by the Quadratic formula that the roots of the quadratic $x^2 - 11x + k$ will only be real when its discriminant $121 - 4k$ is not negative. Clearly, this will only occur when $k \leq 30$. Therefore, 30 of the quadratics in the product have real roots, and it follows that the sum of those real roots is $11 \cdot 30 = \boxed{330}$.

8. The roots of the polynomial $P(x) = x^3 + 5x + 4$ are r, s, and t. Evaluate $(r + s)^4(s + t)^4(t + r)^4$.

Solution: Notice that because $r+s+t=0$, this product is equivalent to $(-r)^4(-s)^4(-t^4) = (rst)^4 = (-4)^4 = \boxed{256}$.

9. Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

Solution: Let $x + y = a$ and let $xy = b$. Then we have that $a^2 + b^2 = 40$ and $a + b = 8$. It follows that $ab = \frac{8^2-40}{2} = 12$. It follows that a and b are roots of the quadratic $z^2 - 8z + 12 = 0$, and therefore, the pair (a, b) is a permutation of $(6, 2)$. It follows that x and y are either roots of the quadratic $z^2 - 6z + 2$ or the roots of the quadratic $z^2 - 2z + 6$. Because the roots of the second quadratic are complex, we must have that x and y are the roots of $z^2 - 6z + 2$. It follows that x is the greater root of this quadratic, or $\frac{6+\sqrt{28}}{2} = \boxed{3 + \sqrt{7}}$.

10. The polynomial $x^4 + ax^3 + bx^2 + cx + d$ has a root at $x = 0$ and a double root at $x = -2$. What is the value of d?

Solution: Because d is the product of the roots and 0 is a root, we must have $d = \boxed{0}$.

11. Find the sum of the squares of the roots of the polynomial $x^2 + 2x - 1$.

Solution: Let the roots be a and b. We know that $a + b = -2$ and $ab = -1$. It follows that $a^2 + b^2 = (a + b)^2 - 2ab = 4 + 2 = \boxed{6}$.

12. Let r_1, r_2, r_3 be the roots of $f(x) = x^3 - 6x^2 - 5x + 22$. Find $\dfrac{1}{r_1r_2} + \dfrac{1}{r_2r_3} + \dfrac{1}{r_3r_1}$.

Solution: Notice that because $r_1r_2r_3 = -22$, we must have that $\frac{1}{r_1r_2} = \frac{1}{\frac{-22}{r_3}} = -\frac{r_3}{22}$ and similar results for the other fractions. Therefore, our answer is $-\frac{1}{22}(r_1 + r_2 + r_3) = \boxed{-\dfrac{3}{11}}$.

13. Let x be the answer to this question. What is the sum of possible values of the average of the numbers $\{-16x, 3x^2, 10x\}$?

Solution: We wish to investigate the average of $3x^2, 10x$, and $-16x$ which is x^2-2x. We are given that the sum of the possible values of this function is equal to x, so we must have $x^2 - 2x = x$. It follows that either $x = 0$, or $x = 3$. Therefore, our answer is $\boxed{3}$.

14. Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find

$$(r + s)^3 + (s + t)^3 + (t + r)^3.$$

Solution: Notice that $r + s + t = 0$. Therefore, we wish to find $-(r^3 + s^3 + t^3)$. Notice that we have that $r^3 = \frac{-1001r-2008}{8}$, and we get similar equations for r and s. Therefore, we wish to find $\frac{1001(r+s+t)+6024}{8} = \frac{6024}{8} = \boxed{753}$.

15. The function $f(x) = x^5 - 20x^4 + ax^3 + bx^2 + cx + 24$ has the interesting property that its roots can be arranged to form an arithmetic sequence. Determine $f(8)$.

Solution: Notice that if the roots are $m - 2d$, $m - d$, m, $m + d$, and $m + 2d$, then the sum of the roots is $5m = 20$, meaning that the middle root is 4. We also have that the product of the roots is $(m^2 - 4d^2)(m^2 - d^2)m = -24$. Plugging in $m = 4$, we get $(16 - 4d^2)(16 - d^2) = -6$. Letting $d^2 = b$, we have that $4b^2 - 80b + 262 = 0$. Solving, we get $b = 10 \pm \sqrt{\frac{69}{2}}$. We wish to calculate $(8 - m + d)(8 - m - d)(8 - m - 2d)(8 - m + 2d)(8 - m) = (4 + d)(4 - d)(4 - 2d)(4 + 2d)4 = 4(16 - d^2)(16 - 4d^2) = 1024 - 320d^2 + 16d^4 = 1024 - 320b + 16b^2$. We can observe that regardless of which value of b we choose, the value of this quadratic will be $269 \cdot 8 - 3200 + 1024 = \boxed{-24}$ as desired.

Note: It was not until after I used this solution that I noticed that because 8 is double the value of the middle term, the sequence $8-m-2d, 8-m-d, 8-m, 8-m+d, 8-m+2d$ is the same as the sequence $m-2d, m-d, m, m+d, m+2d$. It follows that the product of each of the first values is actually the product of the roots, which with Vieta's is trivially $\boxed{-24}$.

3 Sources

1. 2015 AMC 10A Problem 23
2. 2010 AMC 10A Problem 21
3. National Internet Math Olympiad Summer Contest 2017 Problem 9
4. 2009 February Harvard MIT Math Tournament Algebra Problem 5
5. 2015 Purple Comet High School Problem 10
6. 1971 American High School Math Exam Problem 20
7. National Internet Math Olympiad Summer Contest 2013 Problem 4
8. National Internet Math Olympiad Summer Contest 2011 Problem 9
9. 2013 February Harvard MIT Math Tournament Algebra Problem 1
10. 2014 Berkeley Math Tournament Fall Speed Round Problem 38
11. 2014 Berkeley Math Tournament Fall Speed Round Problem 42
12. 2013 Berkeley Math Tournament Fall Speed Round Problem 89
13. 2013 Berkeley Math Tournament Fall Speed Round Problem 90
14. 2015 Berkeley Math Tournament Spring Individual Round Problem 11
15. 2014 Berkeley Math Tournament Spring Analysis Round Problem 4