Problems of the Week with Solutions 2017-2018

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1 Problem of the Week 9/17/17 - 9/23/17

In a standard 6×7 Connect Four Board, tiles are placed in each column in attempts to place four tiles in a row either horizontally, vertically, or diagonally. In how many different positions is it possible to successfully make a row of four tiles? The board below is one position to consider:

–	–	–	–	–	–	–
–	–	–	–	–	–	–
–	–	–	–	■	–	–
–	–	–	■	–	–	–
–	–	■	–	–	–	–
–	■	–	–	–	–	–

Solution: We will do casework on the orientation of the row of four tiles.

Case 1: The row is horizontal.

Notice that every distinct horizontal row has one unique leftmost tile. Therefore, the number of horizontal rows is the same as the number of possible leftmost tiles. The leftmost tile has to be in one of the leftmost four columns as there must be room to place 3 tiles to the right of it. The row of the horizontal row does not matter. Thus, we have 6 ways to choose the row of the leftmost tile and 4 ways to choose the row of the leftmost square, so we have $4 \cdot 6 = 24$ possibilities in this case.

Case 2: The row is vertical.

By similar logic to case 1, each vertical row has a unique top square. There are 3 ways to choose the row of the top square and 7 ways to choose the column of the top square so we have $3 \cdot 7 = 21$ total possibilities for this case.

Case 3: The row is diagonal.

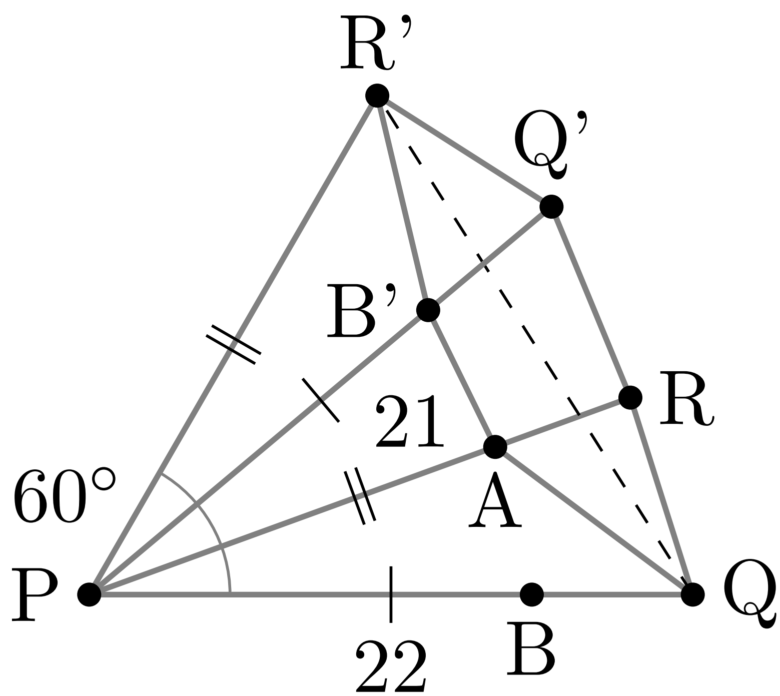
Notice that half of the diagonal rows go from the top left to the bottom right and the others go from the top right to the bottom left. Thus, we will count the number of diagonals which go from the bottom left to the top right and multiply by 2. Each diagonal row of this form has a unique bottom left square. There are 4 ways to choose the column of this square and 3 ways to choose the row of this square, so in total we have $3 \cdot 4 = 12$ possibilities that go from the bottom left to the top right. Thus, there are $2 \cdot 12 = 24$ total rows which are diagonal in this case.

Adding up all of our cases, we have a total of $24+21+24 = \boxed{69}$ different positions where a Connect Four can successfully be made.

Congratulations to **Colin Hannan, Gabe Mogollon, Sam Roth Gordon, Tingting Thompson,** and **Ellie Standifer** for submitting the correct answer!

2 Problem of the Week 9/24/17 - 9/30/17

In $\triangle PQR$, $\overline{PQ} = 22$, $\overline{PR} = 21$, and $\angle QPR = 20°$. Points A and B are chosen on segments $\overline{PR}$ and $\overline{PQ}$, respectively. Compute the square of the minimum possible value of $\overline{QA} + \overline{AB} + \overline{BR}$.



Hint: Use Reflection

Solution: We will proceed by reflection. Reflect point Q across line PR to a point Q'. By definition, if we reflect B across line PR, because B lies on line PQ, the reflection of B, B', will lie on line PQ'. Finally, reflect point R across line PQ' to a point R'. By the definition of these reflections, we have $\overline{AB'} \cong \overline{AB}$ and $\overline{B'R'} \cong \overline{B'R} \cong \overline{BR}$. It follows that the thing we are trying to minimize, $\overline{QA} + \overline{AB} + \overline{BR}$, is equivalent to $\overline{QA} + \overline{AB'} + \overline{B'R'}$. We can notice that the sum of these segments is a path from Q to R'. Therefore, the sum of these segments is minimized when the path is the same as segment $\overline{QR'}$. Therefore, the minimum possible value of $\overline{QA} + \overline{AB'} + \overline{B'R'}$ is the same as the minimum possible value of $\overline{QA} + \overline{AB} + \overline{BR}$ which is equal to the length of $\overline{QR'}$. By the definition of these reflections, $\angle QPR' \cong 3 \times \angle QPR = 60°$ By the Law of Cosines, $(\overline{QR'})^2 = (\overline{PQ})^2 + (\overline{PR'})^2 - 2 \cdot (\overline{PR'}) \cdot (\overline{PQ}) \cdot \cos 60° = 22^2 + 21^2 - 2 \cdot 21 \cdot 22 \cdot \frac{1}{2} = \boxed{463}$ as desired.

Nobody solved this problem of the week.

Note: This problem is very difficult. It is an adapted version of 2014 AMC 12A Problem 20.

3 Problem of the Week 10/1/17 - 10/7/17

Fill in each square of this grid with a number from 1 to 16, using each number exactly once. Numbers to the left and above the grid are the largest sum of two numbers in that row or column. Numbers to the right or below the grid give the largest difference of two numbers in that row or column. We have given you two of the values already, and you are given that the number 7 appears in the top row.

Problem

	27	27	24	22	
29	A				14
28		B		9	12
21		5	C		11
21				D	8
	14	12	10	12	

Solution

	27	27	24	22	
29	7	15	14	1	14
28	16	12	4	9	12
21	2	5	8	13	11
21	11	3	10	6	8
	14	12	10	12	

Translate the numbers in the cells marked A, B, C, and D to the alphabet using the key

$$1 \rightarrow A, 2 \rightarrow B, \cdots 26 \rightarrow Z$$

What is the four-letter result once these numbers have been translated? (The letters should be in the order A, B, C, D, where A is the value of the number in cell A, etc.)

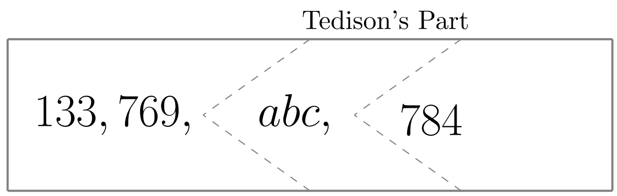
Note: This question is an adapted version of USAMTS $2016-2017$ Round 3 Problem 1.

Solution: Throughout this proof, let (1) refer to the largest sum of the numbers in the row and (2) refer to the largest difference of the numbers in the row. Consider the 1st row. By (1), either the numbers 16 and 13 are in this row or the numbers 15 and 14 are in this row. Consider the 2nd row. By (1), either the numbers 15 and 13 are in this row or the numbers 16 and 12 are in this row. If 15 and 13 were in this row, no possibility would exist for the first row. Therefore, 16 and 12 are in the 2nd row and 15 and 13 are in the 1st row. Consider the first row. By (2), and the fact that 15 is in this row, we must have 1 is in this row. Therefore, it is impossible for both 1 and 15 to be in the 1st column. Consider the 1st column. By (2), 16 and 2 must be in this column. Therefore, 16 is at the intersection of the first column and second row. It follows that 15 and 14 are in the first row, and that 12 is in the second row. With the information given, 12 can either be in the second column or the third column. Consider the third column. By (1), the largest sum is 24. These largest numbers can be written as $12 + x$ and $12 - x$ in order for the numbers to sum to 24. It follows that 12 would have to appear twice in this column to appear in this column. Therefore 12 is at the intersection of the second row and the second column. Now consider the second column. By (1), we know $27 - 12 = 15$ must appear in this column. It follows that 15 is at the intersection of the first row and the second column. We know 14 is in the first row. If it were in the first column, (1) for that column would have to be 30 which it is not. It it were in the fourth column, (1) would have to be at least $14 + 9 = 23 > 22$. Therefore, 14 is at the intersection of the third column and the first row. We know 1 is in the first row. Consider the fourth column. By (2), we know the largest difference is 12. This is only possible if both 1 and 13 are in this column as we know the locations of 14, 15, and 16. Therefore 1 is at the intersection of the first row and the fourth column. Consider the second row. By (2), 4 appears in this row, so 4 is at the intersection of the second row and the third column. Consider the third row. By (2), the largest difference in this row is 11. This is only possible if both 13 and 2 appear in this row. We know 13 is in the fourth column, so 13 is at the intersection of the third row and the fourth column. We know 2 is in the third row. If it were in the third column, (2) for that column would be $14 - 2 = 12 > 10$. It follows that 2 is at the intersection of the first column and the third row. Because we know 7 is in the top row, we must have 7 is at the intersection of the first row and the first column. Consider the third row. By (1), we must have $21 - 13 = 8$ is in this row. It follows that 8 is at the intersection of the third column and the third row. Consider the second column. By (2), we must have $15 - 12 = 3$ is in this column, so 3 is at the intersection of the second column and the fourth row. Consider the third column. By (1), we must have $24 - 14 = 10$ is in this column, so 10 is at the intersection of the third column and the fourth row. Consider the first column. By (1), we must have $27 - 16 = 11$ is in this column, so 11 is at the intersection of the first column and the fourth row. By process of elimination, 6 is at the intersection of the fourth column and the fourth row. Therefore $A = 7, B = 12, C = 8$, and $D = 6$. Translating these with the given key, we get the answer $\boxed{\textbf{GLHF}}$ as desired.

Congratulations to **Tingting Thompson** for submitting the correct answer!

4 Problem of the Week 10/8/17 - 10/14/17

Person W writes his favorite number on a sheet of paper. His favorite number is a multiple of 1001. He then gives his sheet of paper to Ned the Number Theory Master. However, while W passes his sheet of paper, Tedison the majestic eagle cuts out a piece from the middle of the sheet of paper which contains a three-digit number $\overline{abc}$, where a, b, and c are digits. Ned receives what is left of the sheet of paper and is told that the number on the original completed paper is a multiple of 1001 and that a three-digit part from the middle of the number is currently missing. Let $X = \overline{abc}$. Let N be the number of distinct possible values of X and let M be the maximum value of X. Given that Ned the Number Theory Master is perfect, what would he give as the result of the sum $M + N$? The pieces of paper are shown below:

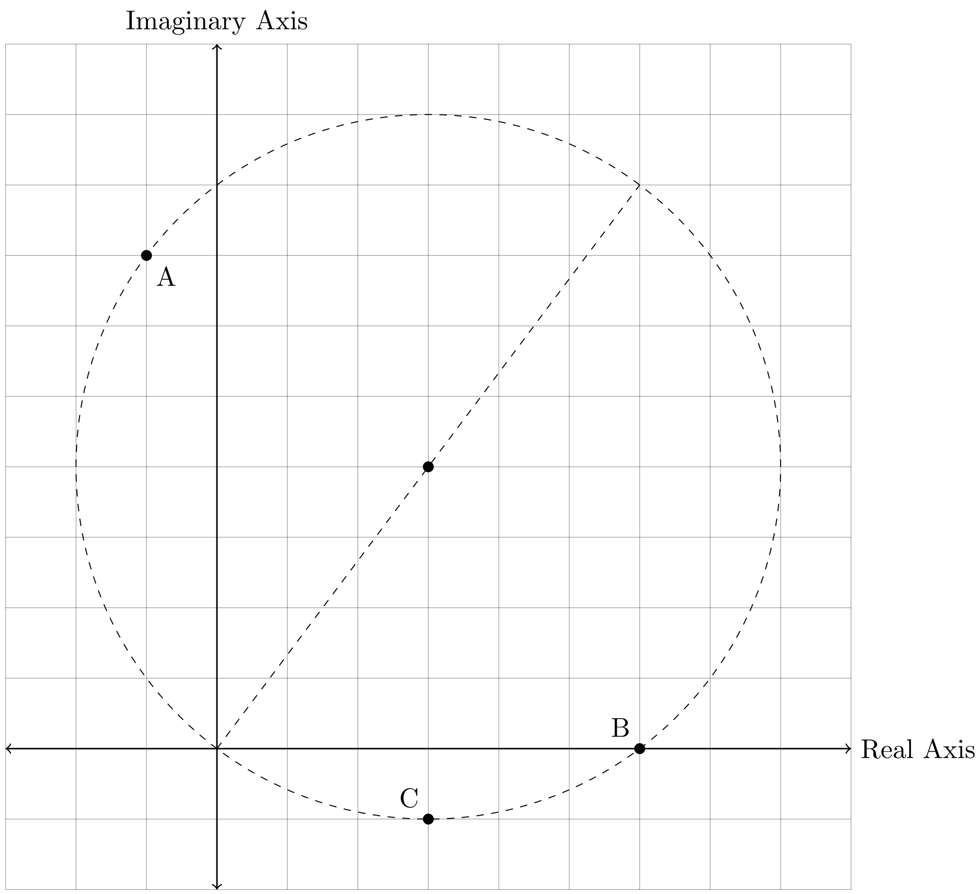


Solution: Notice that this number can be written as $133 \cdot 1000^3 + 769 \cdot 1000^2 + X \cdot 1000 + 784$. We wish to find all values of X such that this number is a multiple of 1001. Notice that $1000^n \equiv (-1)^n \pmod{1001}$. It follows that we wish for $769 + 784 - 133 - X = 1420$ to be a multiple of 1001. Because $0 \le X \le 999$, it follows that the only value of X which works is $1420 - 1001 = 419$. Therefore, $N = 1$ and $M = 419$, and our answer is $419 + 1 = \boxed{420}$ as desired.

Nobody solved this problem of the week.

5 Problem of the Week 10/15/17 - 10/21/17

Consider the complex numbers $A = -1 + 7i$, $B = 6$, and $C = 3 - i$. Consider all complex numbers z such that the quantity $\dfrac{Cz - BC - Az + AB}{Bz - CB - Az + AC}$ is a real number. Determine the unique value of z with this property where the absolute value of z is maximized. Express your answer in the form $a + bi$.



Note: This problem is an adapted version of the 2017 AIME I Problem 10.

Solution: We will begin by proving the following lemma:

Lemma: If $\dfrac{z_3 - z_1}{z_2 - z_1} \cdot \dfrac{z - z_2}{z - z_3}$ is a real number where z_3, z_2, z_1, and z are complex numbers, than z must lie on the circumcircle of the triangle formed by points z_1, z_2, and z_3 in the complex plane.

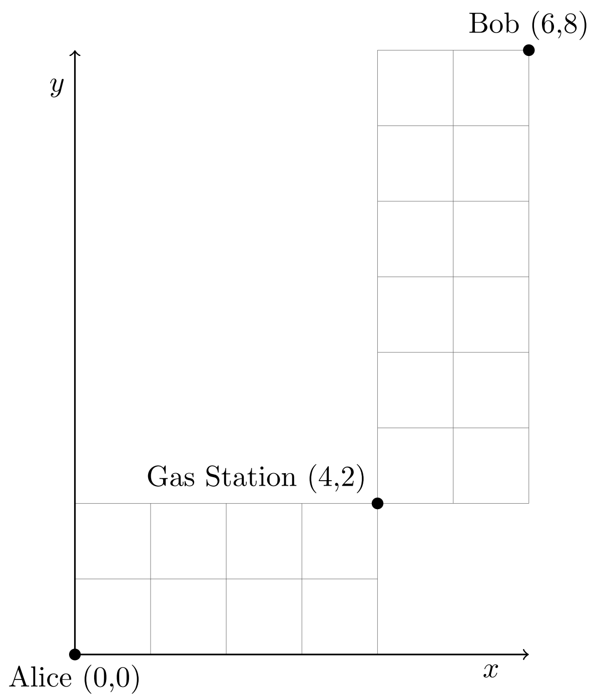
Proof: Notice that if we let $z_3 - z_1 = A$, $z_2 - z_1 = B$, $z - z_2 = C$, and $z - z_3 = D$, then the fraction $\frac{z_3 - z_1}{z_2 - z_1} = \frac{A}{B}$ is a new complex number with an argument that is equal to $\arg(A) - \arg(B)$. (The argument of a complex number is the angle it makes with the positive real axis in the complex plane where the result is an angle between $-\pi$ and π). If we let $\angle z_3 z_1 z_2 = \theta$ in the complex plane, then $\frac{A}{B}$ has an argument of θ. Similarly, if we let $\angle z_2 z z_3 = \alpha$, then $\frac{z - z_2}{z - z_3} = \frac{C}{D}$ has an argument of α. It follows that the product $\frac{AC}{BD}$ is a complex number with an argument of $\alpha + \theta$. Because α and θ are both angles between 0 and π, it follows that $\alpha + \theta = \pi$ if the product is a real number. However, this implies $\angle z_3 z_1 z_2 + \angle z_2 z z_3 = \pi$, which implies $z z_3 z_1 z_2$ is a cyclic quadrilateral as desired.

Now, notice that the fraction $\frac{Cz-BC-Az+AB}{Bz-CB-Az+AC} = \frac{C-A}{B-A} \cdot \frac{z-B}{z-C}$. By the lemma, it follows that if this product is real, z lies on the circumcircle of $\triangle ABC$. It follows that we must find the complex number on the circumcircle of $\triangle ABC$ with the greatest magnitude. By inspection we can note that the complex number $z = 3 + 4i$ satisfies $|A - z| = |B - z| = |C - z| = 5$. However, this point also satisfies $|z| = 5$, so the origin lies on this circle. It follows that we want the reflection of the origin over the complex number $3 + 4i$, and we can easily find that this is $\boxed{6 + 8i}$.

Nobody solved this problem of the week.

6 Problem of the Week 10/22/17-10/28/17

Alice lives at the point $(0, 0)$ in the coordinate plane. She wishes to visit Bob at the point $(6, 8)$ at the coordinate plane. However, before she can visit Bob she must first visit the local gas station at the point $(4, 2)$. Given that she must drive on the roads in the lattice grid shown, and she can only go in the positive x-direction or the positive y-direction (right or up in the diagram below) at any point in time, in how many ways can she make her trip to visit Bob?



Solution: Notice that because Alice's path from her starting point to the Gas Station is independent of her path from the gas station to Bob, the total number of ways for her to reach Bob is the same as the product of the number of ways for her to reach the gas station and the number of ways for her to reach Bob from the gas station. Notice that her path to the gas station consists of four moves to the right and two moves up, so if we let R represent a right move and U represent a move up, then there is a one to one correspondence between paths to the gas station and rearrangements of the letters RRRRUU. Similarly, there is a one to one correspondence between paths from the gas station to Bob and rearrangements of the letters UUUUUURR. The number of rearrangements of RRRRUU is $\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ and the number of rearrangements of UUUUUURR is $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$. Therefore our answer is $28 \cdot 15 = \boxed{420}$.

Congratulations to **Tingting Thompson** for submitting the correct answer!

Problem of the Week 10/22-17 - 10/28/17 Hints

Tuesday Hint 1: Notice that the route Alice takes from her home to the gas station is independent of the route she takes from the Gas Station to Bob. Therefore, we want to find the product of the number of routes to the Gas Station and the number of routes from the Gas Station to Bob.

Thursday Hint 2: Notice that the route Alice takes from her home to the gas station consists of two moves to the north and four moves to the east. Therefore, we can create a one to one correspondence between permutations of the letters NNEEEE and routes from Alice to the gas station.

7 Problem of the Week 10/29/17 - 11/4/17

Determine the sum of all positive integer factors of $80,621,568 = 2^{12} \cdot 3^9$ which are perfect cubes.

Solution: Notice that any positive integer factor of $2^{12} \cdot 3^9$ is of the form $2^a \cdot 3^b$, where $0 \leq a \leq 12$ and $0 \leq b \leq 9$. Also notice that any perfect cube must be of the form $p_1^{3e_1} \cdot p_2^{3e_2} \cdots \cdot p_n^{3e_n}$ where e_i is a sequence of nonnegative integers. It follows that the factors of $2^{12} \cdot 3^9$ which are perfect cubes must be of the form $2^{3a} \cdot 3^{3b}$ where $0 \leq 3a \leq 12$ and $0 \leq 3b \leq 9$. Therefore, we wish to compute

$$2^0 3^0 + 2^3 3^0 + \cdots 2^{12} 3^0 + 2^0 3^3 + 2^3 3^3 + \cdots 2^{12} 3^3 + 2^0 3^6 + 2^3 3^6 + \cdots 2^{12} 3^6 + 2^0 3^9 + 2^3 3^9 + \cdots 2^{12} 3^9$$

By the distributive property, we can factor this as

$$(2^0 + 2^3 + \cdots + 2^{12}) \cdot (3^0 + 3^3 + \cdots + 3^9) = 4681 \cdot 20440 = \boxed{95,679,640}$$

Nobody solved this problem of the week.

8 Problem of the Week 11/5/17 - 11/11/17

Congratulations! You have been selected to play in the game show Python Hall. In this game, Tedison the Brilliant gives you 10 closed doors to choose from. He tells you that there are $\$10,000,000$ behind one of the doors and there is nothing behind each of the other doors. Due to a lack of creativity, you choose the leftmost door without opening it. Tedison carefully observes your choice and decides to open 3 other doors which each contain nothing. He then gives you the choice to switch your choice to one of the other doors. Assuming you decide to switch, your probability of winning the money is now $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $3(m+n)$.

Solution: Notice that if you had stayed with your original choice, your probability of winning would have been $\frac{1}{10}$, as that was the probability of success before Tedison opened one of the doors. Therefore, the probability that one of the other 6 doors contains the money is $1 - \frac{1}{10} = \frac{9}{10}$. Because each of these 6 other doors, the probability of any given door containing the money is $\frac{\frac{9}{10}}{6} = \frac{3}{20}$. Therefore, $m = 3$, and $n = 20$, and $3(m+n) = \boxed{69}$ as desired.

Congratulations to **Joaquin Pesqueira** for submitting the correct answer!

9 Problem of the Week 11/12/17 - 11/18/17

Consider the 5×6 grid shown below. In each blank square, fill in one of the numbers from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the following properties hold:

1. Each of these digits appears exactly 3 times in the grid.

2. A digit in any given square cannot be greater than the digit in the square above it. (They can be equal though.)

3. The sum of the digits in any 2×2 grid of squares must be a multiple of 3.

After this process is completed, let a be the digit in square A, let b be the digit in square B, let c be the digit in square C, and let d be the digit in square D. Compute $1000a + 100b + 10c + d$.

A	7	–	B	–	6
6	–	–	7	–	4
C	3	–	5	–	–
5	–	4	–	1	2
–	1	3	–	D	–

8	7	9	9	9	6
6	3	8	7	8	4
6	3	–	5	–	2
5	1	4	–	1	2
–	1	3	0	0	0

Let the cell in the mth row from the bottom and the nth column from the bottom be denoted as (m, n). Let the 2×2 square consisting of $(m, n), (m, n+1), (m+1, n)$, and $(m+1, n+1)$ be denoted as $(m, n) - (m + 1, n + 1)$ Notice that $(2, 2) \equiv 1 \pmod 3$ as otherwise $(2, 2) - -(3, 3)$ would not have elements which add to a multiple of 3. Also, we know that $(1, 2) \leq (2, 2) \leq (3, 2)$. The only number n such that $n \equiv 1 \pmod 3$ and $1 \leq n \leq 3$ is 1, so we know $(2, 2) = 1$. Similarly, we know $(3, 1) \equiv 0 \pmod 3$ for $(2, 1) - (3, 2)$ to work. It follows that $(3, 1) = 6$. We know $(4, 2) \equiv 0 \pmod 3$ and that $3 \leq (4, 2) \leq 7$. Because 3 6's have been used, we must have $(4, 2) = 3$. We know $(5, 1) \equiv 2 \pmod 3$ and $6 \leq (5, 1)$. It follows that $(5, 1) = 8$. We know that $(3, 3) \equiv 1 \pmod 3$ and that $(3, 3) + (4, 3) \equiv 0 \pmod 3$. It follows that $(4, 3) \equiv 2 \pmod 3$. Similarly, we know $(3, 5) + (3, 6) \equiv 0 \pmod 3$, so we must have $(4, 5) \equiv 2 \pmod 3$. At this point, it is impossible to have 2 9's in the same column, so we know that $(5, 3), (5, 4)$, and $(5, 5)$ are all 9's. Similarly, we know it is impossible for 2 8's to be in the same column, so we know that $(5, 3) = (5, 5) = 8$. It follows that $(3, 5) \equiv 1 \pmod 3$ and that $(2, 4) \equiv 2 \pmod 3$. It follows that it is impossible for 2 0's to be in the same column, so we must have $(1, 4), (1, 5)$, and $(1, 6)$ are all 0's. We also know that $(2, 5) \equiv 1 \pmod 3$, and because it is impossible for another 1 to be in the same column as another 1, it follows that $(5, 2) = 1$. Finally, we must have that $(2, 6) \equiv 2 \pmod 3$, and because $(2, 6) \leq 2$, we must have $(2, 6) = 2$. This is the most we can solve this grid, leaving the squares $(1, 1), (2, 4), (3, 3)$, and $(3, 5)$ unsolvable. It follows that $a = 8$, $b = 9$, $c = 6$, and $d = 0$, so our answer is $\boxed{8960}$.

Note: This problem was inspired by the 2013 – 2014 USAMTS Round 1 Problem 2.

Congratulations to **Joaquin Pesqueira, Tingting Thompson,** and **Nickolas Cosentino** for solving this problem!

10 Problem of the Week 11/19/17 - 11/25/17

In spirit of the 2017 Cryptorally on 11/18/17, I will give you a few messages to decrypt. If you can determine the meaning M of any of these codes, let α be the first letter of M and let β be the last letter of M. Convert α and β to numbers using the conversion $A = 1, B = 2, C = 3, \cdots Z = 26$ and let a and b be the results. Compute $a + b$.

Ciphertext 1: Julius Caesar

VTIGTPSNXIHIXBTIDLXCXRPCIIWXCZDUPLDGSIWPITCSHXCA

Ciphertext 2: A Fine Cipher

HXKVKJJGHNWJLNWGZCSXCAJPVKLVCSHXKGJKGVKM

Ciphertext 3: Rails on the Fence

HINIAANCAKVNEKVAEHSIGRETGB

Solution: We will begin with the first ciphertext. Because the title said Ciphertext 1: Julius Caesar, we can make a safe guess that the first ciphertext was created with a Caesar cipher. A Caesar cipher works by shifting each letter in the plain text some number of characters in the English alphabet, where shifting the letter Z by 1 character results in the letter A. A Caesar cipher can be broken by investigating what letters appear most frequently in the ciphertext and associating those letters with the most common English letters. We can notice that I is very common throughout the cipher text, and it follows that I probably corresponds to one of the most common English letters such as E or T. Indeed, if we let a T in the plain text correspond to an I in the ciphertext, then the resulting ciphertext was created with a Caesar cipher with a shift of 15, resulting in the alphabetical pairing: A - L, B - M, C - N, D - O, E - P, F - Q, G - R, H - S, I - T, J - U, K - V, L - W, M - X, N - Y, O - Z, P - A, Q - B, R - C, S - D, T - E, U - F, V - G, W - H, X - I, Y - J, Z - K. The first letter of each pair corresponds to a letter in the ciphertext while the second letter of each pair corresponds to the corresponding letter in the plaintext. Using this pairing, we can find that the resulting plaintext is GETREADYITSTIMETOWINICANTTHINKOFAWORDTHATENDSINL. It follows that $a = 7$ and $b = 12$, and $a + b = \boxed{19}$.

We can guess that the second ciphertext is an affine cipher due to its title. An affine cipher is similar to a Caesar cipher, but it multiplies each letter in the alphabet by some multiplier before shifting it. Unfortunately, this makes the total number of possible decryption keys $\phi(26) \cdot 26 = 312$, so brute force is much more difficult. However, this cipher can also be broken by frequency analysis. We can notice that J appears frequently throughout the ciphertext, so an E in the plaintext probably corresponds to a J in the ciphertext. Similarly, we can notice that V appears somewhat frequently throughout the ciphertext, and from here we might be able to guess that an I in the plaintext corresponds to a V in the ciphertext. This means a numerical value of 4 corresponds to a numerical value of 9 and a numerical value of 8 corresponds to a numerical value of 21. It follows that if we let the multiplier be a and the shifter be b, then $4a + b \equiv 9 \pmod{26}$ and $8a + b \equiv 21 \pmod{26}$. Subtracting the two equations, we find that $4a \equiv 12 \pmod{26}$ or $4a \equiv 12 \pmod{13}$, and it follows that $a \equiv 3 \pmod{13}$. From here, we can guess that $a = 3$ and $b = 23$, and decoding the ciphertext with these keys, we get a plaintext of MANINEEDMOREWORDSTHATBEGINWITHMANDENDINF. It follows that $a = 13$ and $b = 6$, and $a + b = \boxed{19}$.

We can guess that the last ciphertext is a railfence cipher. A railfence cipher operates by taking plaintext and arranging it diagonally down and then up repeatedly along a set of rails. The key for a railfence cipher is the number of rails. Once the plaintext is arranged, the rails are then read from left to right to produce the ciphertext. Guessing and checking, we can find that a key of 4 rails produces the given ciphertext with the following arrangement:

$H - - - - - I - - - - - N - - - - - I - - - - - A$

$-A - - - N - C - - - A - K - - - V - N - - - E - K$

$- - V - A - - - E - H - - - S - I - - - G - R$

$- - -E - - - - - T - - - - - G - - - - - B$

It follows that $a = 8$ and $b = 11$, and therefore $a + b = \boxed{19}$.

Congratulations to **Orlando Rodriguez** for solving all three of the ciphers in this Problem of the Week!

11 Problem of the Week 11/26/17 - 12/2/17

Let the polynomial $P(x)$ be of the form $P(x) = \pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x^1 \pm 1$. Given that $P(2) = 69$, let $P(3) = A$. Let M be the maximum value of A, and let m be the minimum value of A across all polynomials P with these properties. Compute $2M - m$.

Solution: Notice that when $x = 2$, the expression $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x^1 \pm 1$ can be expressed instead as

$$\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x^1 \pm 1 = x^6 + x^5 + x^4 + x^3 + x^2 + x^1 + 1 - \sum x^{n+1}$$

where the powers of the form x^n were the powers of x that were subtracted in the original expression. When $x = 2$, we can notice that $\sum x^{n+1}$ will produce a unique even integer between 0 and 254 for each polynomial due to the fact that $\sum x^n$ is the binary representation of an integer between 0 and 127. It follows that there is exactly one polynomial of this form where $P(2) = 69$, and therefore $M = m$. Now, by guessing and checking or using the way we manipulated the expression above, we can find that the polynomial $P(x) = x^6 + x^5 - x^4 - x^3 - x^2 + x - 1$ works. It follows that the only possible value of a is $729 + 243 - 81 - 27 - 9 + 3 - 1 = 857$. Therefore, $M = m = 857$, and it follows that $2M - m = \boxed{857}$ as desired.

Congratulations to **Christopher Lehman** for submitting the correct answer!

12 Problem of the Week 12/3/17 - 12/9/17

Over the years, Orlando has acquired a large positive integer number of Cathode Ray Tubes (CRTs). Orlando notices that when he divides his CRTs into piles of 73, there are 33 CRTs left over. Orlando also notices that when he divides his CRTs into piles of 11, there are 8 CRTs left over. Given that Orlando has less than $73 \cdot 11 = 803$ CRTs, how many CRTs does Orlando have?

Solution: Notice that this question is equivalent to finding the smallest number x such that $x \equiv 33 \pmod{73}$ and such that $x \equiv 8 \pmod{11}$. In other words, there exist integers y and z such that $x = 73y + 33 = 11z + 8$. By the Chinese Remainder Theorem (Another common phrase with an acronym of CRT), we know such a value of x exists. It follows that $7y \equiv 8 \pmod{11}$. Noticing that $7 \cdot 8 \equiv 1 \pmod{11}$, it follows that $7y \cdot 8 = 56y \equiv y \equiv 8 \cdot 8 \equiv 9 \pmod{11}$. Therefore, the minimum value of y is 9, and therefore the minimum value of x is $73 \cdot 9 + 33 = \boxed{690}$.

Congratulations to **Christopher Lehman, Orlando Rodriguez, Grace Driskill** and **Tingting Thompson** for submitting the correct answer!

13 Problem of the Week 12/10/17 - 12/16/17

Consider the set of all integer-sided right triangles. Determine the sum of the perimeters of all right triangles among this set which satisfy the property that their perimeter is half of their area, where congruent triangles are only included once. (For example, a $3-4-5$ right triangle is the same as a $4-3-5$ right triangle and if it satisfied the property, it should only be included once in the sum).

Solution: Consider a right triangle with legs of length a and b. If the right triangle satisfies this property, then we must have $a+b+\sqrt{a^2+b^2}=\frac{ab}{4}$. Let $a+b=x$ and let $ab=y$. Then we have $x+\sqrt{x^2-2y}=\frac{y}{4}$. Rearranging, we get $x^2-2y=(x-\frac{y}{4})^2$. Expanding and simplifying, we get $\frac{y^2}{16}=y(\frac{x}{2}-2)$. Because y is greater than 0, we can divide both sides to get $\frac{y}{16}=\frac{x}{2}-2$. Simplifying, we get $y=8x-32$. Now, substituting our definitions for x and y into this equation gives us $ab=8a+8b-32$. Using Simon's Favorite Factoring Trick (SFFT), it follows that $(a-8)(b-8)=32$. Because a and b are integers, it follows that the only pairs a,b which work where $a\leq b$ are $(a,b)=(9,40),(10,24)$, and $(12,16)$. These pairs produce a $9-40-41$ right triangle, a $10-24-26$ right triangle, and a $12-16-20$ right triangle, respectively. It follows that our answer is $9+40+41+10+24+26+12+16+20=\boxed{198}$ as desired.

Nobody solved this problem of the week.

14 Problem of the Week 12/17/17 - 12/23/17

Sir Lenny Face decided to take a break from the emoticon world to travel to an alternate universe. In this alternate universe, the holiday of Christmas has been replaced with the version of Christmas from the song "Twelve Days of Christmas". Essentially, on each day Lenny receives a number of his favorite presents from his true love. If we list his favorite presents as $\{p_1, p_2, p_3, \cdots p_n\}$, then Lenny will receive presents on the n days of Christmas. On the first day of Christmas, his true love will give him 1 p_1. On the second day of Christmas, his true love will give him 1 p_1 and 2 p_2's. On the third day of Christmas, his true love will give him 1 p_1, 2 p_2's, and 3 p_3's, and so on, where on day x, Lenny will receive i p_i's for each i in the range $1 \leq i \leq x$. Let $f(n)$ be the total number of presents Lenny will receive when there are n days of Christmas in this problem. Compute $f(12) + f(5) + f(4) + f(1)$.

Solution: By Gauss Sums, we know that the sum of the first n positive integers is

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

It follows that the generalized version of this question is computing

$$f(n) = \sum_{x=1}^{n} \sum_{y=1}^{x} y = \sum_{x=1}^{n} \frac{x(x+1)}{2} = \sum_{x=1}^{n} \binom{x+1}{2}$$

However, by the Hockey Stick Identity, we can easily simplify this as

$$f(n) = \sum_{x=1}^{n} \binom{x+1}{2} = \binom{n+2}{3}$$

It follows that our answer is $\binom{14}{3} + \binom{7}{3} + \binom{6}{3} + \binom{3}{3} = 364 + 35 + 20 + 1 = \boxed{420}$.

Congratulations to **Carol Lu** and **Christopher Lehman** for submitting the correct answer!

15 Problem of the Week 12/24/17 - 12/30/17

In the history of mankind, there has only ever been one math problem with an answer of $20,170,002,029$. Let a be the month that this problem was published (January is 1, February is 2, etc.) and let b be the year that this problem was published. Compute $10000001b + a$.

Solution: Because the number $20,170,002,029$ is such a weird number and we are given that it has only occurred one time, it makes sense to assume the number was the answer to a recent problem. Because the number 2017 occurs in the number and the year when this problem was posted was 2017, it makes sense to assume the problem was published in 2017. However, we can notice that if the problem was published in December of 2017, then $b = 2017$ and $a = 12$, and our answer would be $10000001 \cdot 2017 + 12 = 20,170,002,029$. Because this problem was posted in December of 2017, it follows that the problem is referring to itself. Therefore our answer is $\boxed{20,170,002,029}$.

Congratulations to **Tingting Thompson** and **Carol Lu** for submitting the correct answer!

16 Problem of the Week 12/31/17 - 1/6/18

Consider the number $N = 2018^{2017}$. Let x be the number of triples of positive integers (a, b, c) with the property that $abc = N$. Compute $\sqrt{x}$.

Solution: Notice that the prime factorization of 2018 is $2018 = 2^1 \cdot 1009^1$. Therefore, we can write any triple (a, b, c) in the form $a = 2^{a_1}1009^{a_2}, b = 2^{b_1}1009^{b_2}$, and $c = 2^{c_1}1009^{c_2}$ where $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = 2017$. Therefore, the problem is equivalent to finding the number of ordered 6-tuples $(a_1, a_2, b_1, b_2, c_1, c_2)$ such that $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = 2017$. By Stars and Bars, the number of ways to choose nonnegative integers a_1, b_1, and c_1 such that $a_1 + b_1 + c_1 = 2017$ is $\binom{2019}{2}$. Similarly, the number of ways to choose a_2, b_2, and c_2 is $\binom{2019}{2}$. It follows that $x = \binom{2019}{2}^2$ and $\sqrt{x} = \binom{2019}{2} = 2019 \cdot 1009 = \boxed{2,037,171}$ as desired.

Congratulations to **Carol Lu** for submitting the correct answer!

17 Problem of the Week 1/7/18 - 1/13/18

Consider the 7th-degree polynomial $P(x) = x^7 - x^6 + 2x + 1$. Let its roots be $a_1, a_2, a_3, a_4, a_5, a_6$, and a_7. Compute $(2 - a_1)(2 - a_2)(2 - a_3)(2 - a_4)(2 - a_5)(2 - a_6)(2 - a_7)$.

Solution: Remembering that a polynomial of degree n in x with a leading term of ax^n can be factored as $a \cdot \prod_{i=1}^{n}(x - r_i)$ where r_i is the ith root, we can notice that the desired product is simply $P(2)$. We can easily compute this to be $2^7 - 2^6 + 2 \cdot 2 + 1 = \boxed{69}$.

Congratulations to **Carol Lu** for submitting the correct answer!

18 Problem of the Week 1/14/18 - 1/20/18

Consider the vertices of a regular 2018-gon with 2018 equal sides and 2018 equal angles. Determine the number of rectangles which can be formed by connecting 4 of the vertices of this regular 2018-gon.

Hint: Try investigating this problem for smaller numbers of vertices such as for a regular 4-gon or a regular 6-gon. Do you notice anything about each half of the vertices?

Solution: Consider any set of 1009 consecutive vertices of this regular 2018-gon. Notice that for any rectangle with all four of its vertices among these 2018 vertices, exactly 2 of the vertices of that rectangle must be among our set of 1009 consecutive vertices, as otherwise 3 of the vertices of the rectangle would lie on a set of 1009 consecutive vertices and therefore the triangle formed by them would be an obtuse triangle which is impossible for the vertices of a rectangle. Similarly, we can notice that for any set of 2 vertices among our set of 1009 consecutive vertices, we can form exactly one unique rectangle by reflecting each vertex to the opposite vertex on the regular 2018-gon to produce other vertices of a rectangle. It follows that the total number of rectangles which can be formed is $\binom{2019}{2} = \boxed{2,037,171}$.

Congratulations to **Carol Lu** for submitting the correct answer!

19 Problem of the Week 1/21/18 - 1/27/28

There are five numbers N of the form $N = \overline{abcdefghij}$ (where letters $a, b, c, d, e, f, g, h, i,$ and j are digits) such that $N = \overline{abcdefghij} = (\overline{abcde})^2 + (\overline{fghij})^2$ and $a > 0$. Given that the only pairs of positive integers (k, l) such that $k^2 + l^2 = 10000000001 = 100000^2 + 1$ and $k < l$ are $(k, l) = (1, 100000), (19801, 98020), (48320, 87551)$, and $(64700, 76249)$, find the smallest 10-digit integer N with these properties.

Note: While you are permitted to use a computer or a calculator for this question, it is possible to complete this question without a computer or a calculator with the extra information given.

Solution: Let $\overline{abcde} = x$ and let $\overline{fghij} = y$. Then the given equation is equivalent to $100000x+y = x^2 + y^2$. Rearranging this equation, we get $y^2 - y + x^2 - 100000x = 0$. Using the quadratic formula on this equation in terms of y, we get $y = \frac{1 \pm \sqrt{1+400000x-4x^2}}{2}$. In order for this to be an integer, we must have $1 + 400000x - 4x^2 = z^2$ for some integer z. Rearranging this equation, we get $z^2 + (2x - 100000)^2 = 100000^2 + 1$. It follows that the pair $(z, 2x - 100000)$ must be among the pairs (k, l) which were listed in the problem. Because $2x - 100000$ is clearly even, and we wish to minimize x, we want $2x - 100000$ to be the minimum even integer which corresponds to one of these pairs. Clearly x will be minimized when $2x - 100000 = -64700$, which corresponds to $x = 17650$. Plugging this into the equation from using the quadratic formula, we get $y = 38125$. It follows that the minimum value of N is $\boxed{1,765,038,125}$.

Nobody solved this problem of the week.

20 Problem of the Week 1/28/18 - 2/3/18

Call an integer z *fat* if $1 \mid z$, $2 \mid (z+1)$, $3 \mid (z+2)$, $4 \mid (z+3)$, and $5 \mid (z+4)$. Compute the number of *fat* integers z in the range $1 \leq z \leq 7200$.

Note: The notation $x \mid y$ means that x divides y, or the result when y is divided by x is an integer.

Notice that for any *fat* integer z, we must have $z \cong 0 \pmod 1$, $z \cong -1 \pmod 2$, $z \cong -2 \pmod 3$, $z \cong -3 \pmod 4$, and $z \cong -4 \pmod 5$. It follows by the Chinese Remainder Theorem that $z \cong 1 \pmod{60}$, as $\operatorname{lcm}(1, 2, 3, 4, 5) = 60$. Therefore, for any range of 60 integers, there is exactly one *fat* integer, and therefore the total number of *fat* integers is $\frac{7200}{60} = \boxed{120}$.

Congratulations to **Ernesto Alcala** and **Carol Lu** for submitting the correct answer!

21 Problem of the Week 2/4/18 - 2/10/18

5 athletes named Express, Focus, Gold, Wave, and Nick compete in a race. In how many different ways can they finish given that ties between two or more of the athletes are possible?

Note: Two rankings are considered different if two athletes in one ranking are placed in a different way than in the other ranking. Simply changing the order of two teams that are tied in one ranking does not change the ranking. For example, there is only one unique ranking in which all 5 of the athletes tie.

Solution: We will break up the rankings into the following cases:

Case 1: Nobody ties.

Clearly in this case there are $5! = 120$ possibilities.

Case 2: Two people tie and the rest do not.

There are $\binom{5}{2} = 10$ ways to choose which two athletes tie and then $4! = 24$ ways to order the four finishing groups for a total of $10 \cdot 24 = 240$ possibilities.

Case 3: Three people tie and the others do not.

There are $\binom{5}{3} = 10$ ways to choose who ties and $3! = 6$ ways to order the ranks for a total of 60 possibilities.

Case 4: Four people tie.

There are $\binom{5}{1} = 5$ ways to choose who doesn't tie and $2! = 2$ ways to order the ranks for a total of 10 possibilities.

Case 5: Five people tie.

Clearly there is 1 possibility in this case.

Case 6: Two different pairs of two people tie.

There are $\binom{5}{1} = 5$ ways to choose who doesn't tie, $\binom{3}{1} = 3$ ways to choose who ties with an arbitrary remaining athlete, and $3! = 6$ ways to order the ranks for a total of $5 \cdot 3 \cdot 6 = 90$ possibilities.

Case 7: Three people tie and the other two people also tie.

There are $\binom{5}{2} = 10$ ways to choose which pair ties and $2! = 2$ ways to order the ranks for a total of 20 possibilities.

Adding up all of our cases, we get an answer of $120 + 240 + 60 + 10 + 1 + 90 + 20 = \boxed{541}$.

Congratulations to **Carol Lu** for submitting the correct answer!

22 Problem of the Week 2/11/18 - 2/17/18

Given that the Basel Sum or the sum of the reciprocals of the squares of the natural numbers is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$, compute the sum

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

which is the sum of the reciprocals of the squares of all the natural numbers which are not divisible by 2 or 3.

Solution: Instead of directly computing the sum, we will instead find the sum of the reciprocals of the squares of all natural numbers which are divisible by 2 or 3 and subtract that from the original sum. The sum of the reciprocals of the squares of all natural numbers which are divisible by 2 is

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{2^2} \cdot \frac{1}{n^2} = \frac{1}{4} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{24}$$

Similarly, the sum of the reciprocals of the squares of all natural numbers which are divisible by 3 is

$$\sum_{n=1}^{\infty} \frac{1}{(3n)^2} = \frac{1}{3^2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{54}$$

By adding these two sums, we would get the sum of the reciprocals of the squares of all natural numbers which are divisible by 2 or 3, but we would include the reciprocals of the squares of all natural numbers which are divisible by 6 twice. Therefore, we need to subtract the following sum which is the sum of the reciprocals of the squares of all natural numbers which are divisible by 6:

$$\sum_{n=1}^{\infty} \frac{1}{(6n)^2} = \frac{1}{36} \cdot \frac{\pi^2}{6} = 216$$

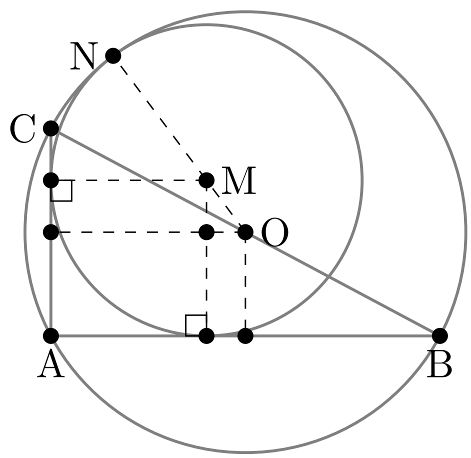
It follows that the sum of the reciprocals of the squares of all natural numbers which are divisible by 2 or 3 is $\frac{\pi^2}{24} + \frac{\pi^2}{54} - \frac{\pi^2}{216} = \frac{\pi^2}{18}$. It follows that our answer is the difference between the original Basel Sum and this sum, or $\frac{\pi^2}{6} - \frac{\pi^2}{18} = \boxed{\frac{\pi^2}{9}}$.

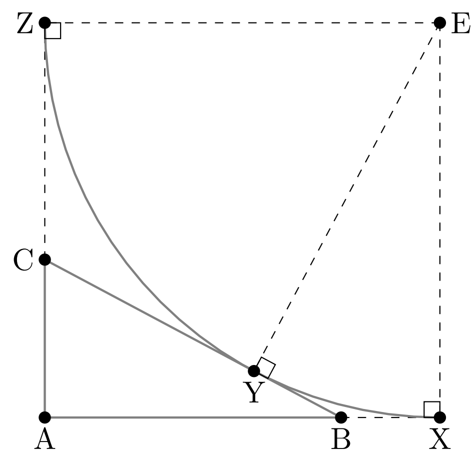
Congratulations to **Maxwell Thum**, **Ernesto Alcala**, and **Carol Lu** for submitting the correct answer!

23 Problem of the Week 2/18/18 - 2/24/18

Consider $\triangle ABC$ with $\overline{AB} = 15$, $\overline{AC} = 8$, and $\overline{CB} = 17$. Let O be the center of the *circumcircle* of $\triangle ABC$, or in other words, let O be the point such that $\overline{AO} \cong \overline{OC} \cong \overline{OB}$ and define the *circumcircle* of $\triangle ABC$ as the circle centered at O which passes through A, B, and C. Define the *mixtilinear incircle* of $\angle BAC$ of $\triangle ABC$ as the circle which is tangent to sides $\overline{AC}$ and $\overline{AB}$ of $\triangle ABC$ and is also tangent to the *circumcircle* of $\triangle ABC$. Let M be the center of the *mixtilinear incircle* of $\angle BAC$. Define the *excircle* of $\angle BAC$ of $\triangle ABC$ as the circle which is tangent to the extensions of sides $\overline{AB}$ and $\overline{AC}$ and is also tangent to side $\overline{BC}$ of $\triangle ABC$. Let E be the center of the *excircle* of $\angle BAC$ of $\triangle ABC$. Determine the length of segment $\overline{ME}$.

Note: This problem was inspired by problem 21 on the 2018 AMC 12B.





Solution: We will proceed with coordinates. Let A be the origin at $(0, 0)$, let B be the point $(15, 0)$, and let C be the point $(0, 8)$. By the Pythagorean Theorem, we know that we can do this as $\triangle ABC$ is a right triangle. It is well known that the circumcenter of a right triangle is at the midpoint of its hypotenuse, so we know that O is at $(7.5, 4)$. If we let the radius of the mixtillinear incircle be r, then we know that M is at (r, r). Now let $\vec{OM}$ intersect the circumcircle of $\triangle ABC$ and the mixtillinear incircle in question at point N. We know that $\overline{ON} = \frac{\overline{BC}}{2} = 8.5$, so we know that $\overline{OM} = 8.5 - r$. However, by the Distance Formula we also know that $\overline{OM} = \sqrt{(r-4)^2 + (r-7.5)^2}$. It follows that $(8.5 - r)^2 = (r-4)^2 + (r-7.5)^2$. Solving, we get $r = 6$, and therefore the coordinates of M are $(6, 6)$. Now let the foot of the perpendicular from E to $\overline{AB}$ be X, let the foot of the perpendicular from E to $\overline{BC}$ be Y, and let the foot of the perpendicular from E to $\overline{AC}$ be Z and let the radius of the excircle be R. We know that $\overline{BX} = R - 15$ and we know that $\overline{ZC} = R - 8$. However, by Power of a Point we also know that $\overline{ZC} = \overline{YC}$ and that $\overline{XB} = \overline{YB}$. Therefore $\overline{BC} = 17 = 2R - 23$, and solving we get $R = 20$. Therefore the coordinates of E are $(20, 20)$. It follows by the distance formula that the length of segment $\overline{ME}$ is $\sqrt{2 \cdot (20-14)^2} = \boxed{14\sqrt{2}}$.

Nobody solved this problem of the week.

24 Problem of the Week 2/25/18 - 3/3/18

Let the roots of the polynomial $x^4 - 20x^3 + 49x^2 - 840x + 2$ be a, b, c, and d. Compute

$$\frac{1}{20a^3 - 49a^2 + 840a - 2} + \frac{1}{20b^3 - 49b^2 + 840b - 2} + \frac{1}{20c^3 - 49c^2 + 840c - 2} + \frac{1}{20d^3 - 49d^2 + 840d - 2}$$

Solution: Notice that by definition

$$a^4 - 20a^3 + 49a^2 - 840a + 2 = 0 \to a^4 = 20a^3 - 49a^2 + 840a - 2$$

We can find a similar result for b, c, and d. It follows that we wish to calculate $\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4}$. It is well known that if $r_1, r_2, r_3, \cdots r_n$ are roots of the polynomial $o(x) = m_n x^n + m_{n-1}x^{n-1} + \cdots + m_1 x^1 + m_0 x^0 = 0$, then $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \cdots \frac{1}{r_n}$ are the roots of the *reciprocal polynomial* of $o(n)$, $p(n) = r_0 x^n + r_1 x^{n-1} + \cdots + r_{n-1}x^1 + r_n x^0 = 0$ which is formed by reversing the coefficients of $o(n)$. Therefore, if we let $e = \frac{1}{a}, f = \frac{1}{b}, g = \frac{1}{c}$, and $h = \frac{1}{d}$, then e, f, g, and h are the roots of the polynomial $2x^4 - 840x^3 + 49x^2 - 20x + 1 = 0$, and we wish to find $e^4 + f^4 + g^4 + h^4$. Through brute force we can find that this can be rewritten as $e^4 + f^4 + g^4 + h^4 =$

$$((e+f+g+h)^2 - 2(ef+eg+eh+fg+fh+gh))^2 - 4efgh + 4(e+f+g+h)(efg+efh+egh+fgh)$$

By Vieta's Formulas, we know that $e + f + g + h = \frac{840}{2} = 420$, $ef + eg + eh + fg + fh + gh = \frac{49}{2}$, $efg + efh + egh + fgh = \frac{20}{2} = 10$, and $efgh = \frac{1}{2}$. It follows that our answer is

$$(420^2 - 49)^2 - 2 \cdot \frac{49^2}{4} - 4 \cdot \frac{1}{2} + 4 \cdot 420 \cdot 10 = \boxed{\frac{62199381597}{2}}$$

Congratulations to **Carol Lu** for submitting the correct answer!

25 Problem of the Week 3/4/18 - 3/10/18

Consider all 4-element subsets of the 15-element set $\{1, 2, 3, \cdots 14, 15\}$. For example, our subset could be $\{1, 2, 3, 4\}$ or $\{4, 7, 9, 12\}$. Let A be the average value of all the maximum elements of these 4-element subsets. Let $A = \frac{m}{n}$ for two relatively prime positive integers m and n. Compute $m + n$.

Solution: Notice that the average value is equivalent to the sum of the maximum elements of all 4-element subsets divided by the total number of subsets. The number of 4-elements subsets with a maximum element of 15 is $\binom{14}{3}$ as the other three elements must be less than 15. The number of 4-element subsets with a maximum element of 14 is $\binom{13}{3}$. Following this pattern, we find that when $n \geq 4$, the number of 4-element subsets with a maximum element of n is $\binom{n-1}{3}$. It follows that the sum of the maximum elements of all 4-element subsets is

$$\sum_{n=4}^{15} n \cdot \binom{n-1}{3} = 4\binom{3}{3} + 5\binom{4}{3} + 6\binom{5}{3} + \cdots + 15\binom{14}{3}$$

Notice that this sum is equivalent to

$$\sum_{i=1}^{12} (16 - i) \cdot \binom{15-i}{3} = 16 \cdot \left(\sum_{i=1}^{12} \binom{15-i}{3} \right) - \left(\sum_{i=1}^{12} i \cdot \binom{15-i}{3} \right)$$

The first sum in this expansion, $\sum_{i=1}^{12} \binom{15-i}{3} = \binom{14}{3} + \binom{13}{3} + \cdots + \binom{3}{3}$ can be quickly simplified using the Hockey Stick Identity. The Hockey Stick Identity states that for any n and any k, we must have $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$. It follows that

$$\sum_{i=1}^{12} \binom{15-i}{3} = \sum_{k=3}^{14} \binom{k}{3} = \binom{15}{4}$$

The second term in our sum, $\sum_{i=1}^{12} i \cdot \binom{15-i}{3} = \binom{14}{3} + 2\binom{13}{3} + 3\binom{12}{3} + \cdots + 12\binom{3}{3}$ can be rewritten as

$$\sum_{i=1}^{12} i \cdot \binom{15-i}{3} = \sum_{o=3}^{14} \left(\sum_{p=3}^{o} \binom{p}{3} \right)$$

Using the Hockey Stick Identity, this new term can be rewritten as

$$\sum_{o=3}^{14} \left(\sum_{p=3}^{o} \binom{p}{3} \right) = \sum_{o=3}^{14} \binom{o+1}{4} = \binom{16}{5}$$

It follows that the sum of all the maximum elements is $16\binom{15}{4} - \binom{16}{5}$. The total number of 4-element subsets is $\binom{15}{4}$. Therefore, the average value is

$$\frac{16\binom{15}{4} - \binom{16}{5}}{\binom{15}{4}} = 16 - \frac{\frac{16!}{5! \cdot 11!}}{\frac{15!}{4! \cdot 11!}} = 16 - \frac{16}{5} = \frac{64}{5}$$

It follows that our answer is $64 + 5 = \boxed{69}$.

Congratulations to **Carol Lu** for submitting the correct answer!

26 Problem of the Week 3/11/18 - 3/17/18

Sigma and Phi are playing a game. In this game, they will take turns performing operations on some arbitrary number N and then giving the result to their opponent. When N is even, the player has the option to give their opponent the number $\frac{N}{2}$. When N is divisible by 4, the player also has the option to give their opponent the number $\frac{N}{4}$. When N is odd, the player can either give their opponent the number $\frac{N-1}{2}$ or the number $\lfloor \frac{N}{4} \rfloor$. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.) The first player to give their opponent the number 0 wins the game.

For example, if Sigma began the game with the number 12, he could either give Phi the number 6 or the number 3. If Sigma gave Phi the number 3, then because 3 is odd, Phi could either give Sigma the number 1 or the number 0. If Sigma gave Phi the number 6, then because 6 is even, Phi could only give Sigma the number 3. In this case, if both players played perfectly, then Sigma would win in exactly 3 total moves (including Phi's moves).

Given that Sigma and Phi play perfectly and Sigma begins the game by performing an operation on the number $N = 2^{665} + 2^{664} + 2^{210}$, then the total number of moves can be Y (This includes all of Sigma's moves and all of Phi's moves.) What is the sum of all possible values of Y?

Solution: Notice that the transformation $N \to \dfrac{N}{2}$ is equivalent to removing the last digit in the binary representation of N. The transformation $N \to \dfrac{N}{4}$ is equivalent to removing the last two digits in the binary representation of N. The transformation $N \to \dfrac{N-1}{2}$ is equivalent to removing the last digit of the binary representation N when N is odd, and the transformation $N \to \lfloor \dfrac{N}{4} \rfloor$ is equivalent to removing the last 2 digits of the binary representation of N. With these facts in mind, we can think of the entire game based on the binary representation of N. The binary representation of $N = 2^{665} + 2^{664} + 2^{210}$ is

$$N = 11000\cdots00100\cdots00_2$$

where the first block of 0's contains 453 0's and the second block of 0's contains 210 0's. We claim that Phi can always win the game regardless of what Sigma does. Consider all of the moves before the binary representation of N becomes odd. If Phi always does the opposite move of what Sigma does (for example, when Sigma divides N by 4, Phi divides N by 2), then Phi can consistently eliminate blocks of 3 0's from the end of the binary representation of N until N becomes odd. This would leave Sigma with the new value of $N = 2^{455} + 2^{454} + 1$. The binary representation of this number is

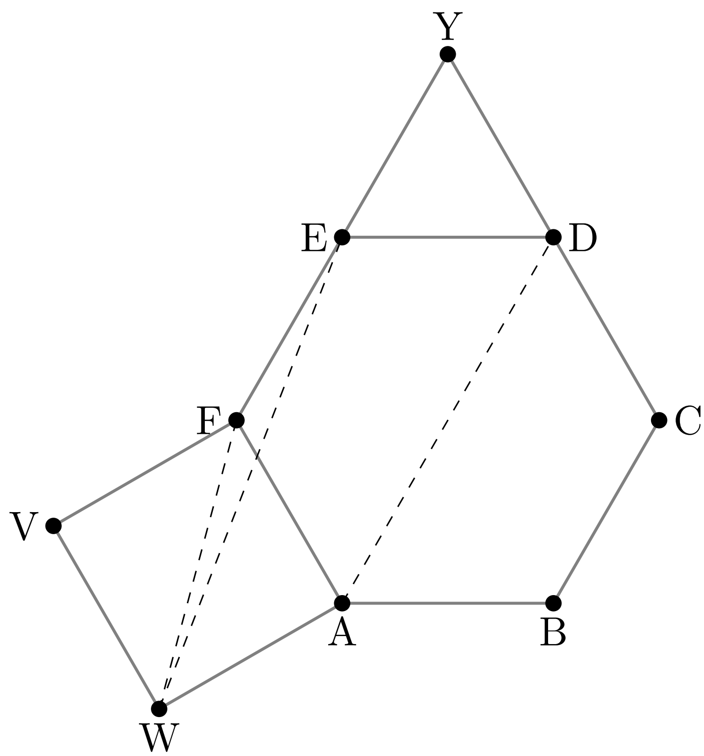
$$N = 1100\cdots001_2$$

where the block of 0's contains 453 0's. From this point, Sigma can either leave Phi with the number $N = 2^{454} + 2^{453}$ which is $1100\cdots00_2$ with 453 0's or the number $N = 2^{453} + 2^{452}$ which is $1100\cdots00_2$ with 452 0's. In either case, Phi can respond by returning the number $N = 2^{452} + 2^{451}$ with 451 0's. From here, Phi can keep doing the strategy of performing the opposite move of what Sigma performs and removing blocks of 3 0's from the end of the binary representation until Sigma is left with $N = 2^2 + 2^1 = 110_2$. From here, Sigma has to return the number $N = 2^1 + 2^0 = 3$, and then Phi can return the number $N = 0$, winning the game. This means that the entire process of the game involves removing 3 digits from the binary representation of N in series of 2 moves. It follows that our answer is $\frac{2}{3} \cdot (665 + 1) = \boxed{444}$.

Congratulations to **Carol Lu** for submitting the correct answer!

27 Problem of the Week 3/18/18 - 3/24/18

Consider regular hexagon $ABCDEF$ with side length 2. The square $AFVW$ is constructed on side $\overline{AF}$ of hexagon $ABCDEF$ such that points V and W lie outside of $ABCDEF$. Equilateral triangle $\triangle EYD$ is constructed on side $\overline{ED}$ of hexagon $ABCDEF$ such that point Y lies outside of $ABCDEF$. Determine the area of pentagon $DAWEY$.



Solution: We can notice that the area of $DAWEY$ is equivalent to the difference between the area of pentagon $WFYDA$ and triangle $\triangle FEW$. The area of pentagon $WFYDA$ is trivially $4 \cdot \frac{2^2\sqrt{3}}{4} + \frac{2 \cdot 2}{2} = 2 + 4\sqrt{3}$. We know that $\angle EFW = 90° + 120° - \angle VFW = 210° - 45° = 165°$. It follows that the area of $\triangle FEW$ is $\frac{2 \cdot 2\sqrt{2} \cdot \sin 165°}{2} = 2\sqrt{2} \cdot \frac{\sqrt{6}-\sqrt{2}}{4} = \sqrt{3} - 1$. Therefore, our answer is $2 + 4\sqrt{3} - (\sqrt{3} - 1) = \boxed{3 + 3\sqrt{3}}$.

Congratulations to **Carol Lu** for submitting the correct answer!

28 Problem of the Week 3/25/18 - 3/31/18

Tedison randomly selects two squares on a regular 8×8 chessboard without replacement. Given that the two squares Tedison selected do not share any vertices, what is the probability that Tedison's two squares are either in the same row or in the same column?

Note: The fact that the squares do not share any vertices means that they are not adjacent either horizontally, vertically, or diagonally.

Solution: There are $\binom{64}{2} = \frac{64 \cdot 63}{2} = 2016$ total ways to choose two different squares. The total number of ways to choose two squares which are horizontally adjacent is $8 \cdot 7 = 56$. Similarly, the number of ways to choose two squares which are vertically adjacent is $7 \cdot 8 = 56$. The number of ways to choose two squares which are diagonally adjacent is $2 \cdot 7 \cdot 7 = 98$, as there are 2 ways to choose the orientation of the square and $7 \cdot 7 = 49$ ways to choose the position of the leftmost square. It follows that the total number of ways to choose two squares which are not adjacent is $2016 - 56 - 56 - 98 = 1806$. The number of ways to choose two squares which are in the same row or column but are not adjacent is $16 \cdot (\binom{8}{2} - 7) = 16 \cdot 21 = 336$. It follows that our answer is $\frac{336}{1806} = \boxed{\frac{8}{43}}$.

Congratulations to **Carol Lu** and **Brycklen Arnold** for submitting the correct answer!

29 Problem of the Week 4/1/18 - 4/7/18

Zero the Lagomorph has 10 red Easter eggs, 10 green Easter eggs, and 10 blue Easter eggs. He wishes to arrange 10 of these eggs in a horizontal row such that going from left to right along the row, the following rules are all satisfied:

(1) A blue egg must never be followed by a red egg.

(2) A red egg must never be followed by 2 consecutive green eggs.

(3) A green egg must never be followed by 3 consecutive blue eggs.

Find the number of ways that Zero can create this row of 10 eggs under the above conditions.

Note: All Easter eggs of the same color are indistinguishable.

Solution: Let R represent a Red egg, let G represent a Green egg, and let B represent a Blue egg. Define a_n as the number of ways to arrange n eggs such that the row ends in XR where X is not B. Let b_n be the number of ways to arrange n eggs such that the row ends in XG where X is not R or RG. Let c_n be the number of ways to arrange n eggs such that the row ends in XB where X is not G, GB, or GBB. Let d_n be the number of ways to arrange n eggs such that the row ends in RG. Let e_n be the number of ways to arrange n eggs such that the row ends in GB, and let f_n be the number of ways to arrange n eggs such that the row ends in GBB. Then we have the following recurrence relations:

$$a_n = a_{n-1} + b_{n-1} + d_{n-1}$$

$$b_n = b_{n-1} + c_{n-1} + e_{n-1} + f_{n-1}$$

$$c_n = a_{n-1} + c_{n-1}$$

$$d_n = a_{n-1}$$

$$e_n = b_{n-1} + d_{n-1}$$

$$f_n = e_{n-1}$$

We can easily calculate that $a_1 = 1$, $b_1 = 1$, and $c_1 = 1$, and d_1, e_1, and f_1 are all 0. Using this, we can create the following table of values:

n	a_n	b_n	c_n	d_n	e_n	f_n
1	1	1	1	0	0	0
2	2	2	2	1	1	0
3	5	5	4	2	3	1
4	12	13	9	5	7	3
5	30	32	21	12	18	7
6	74	78	51	30	44	18
7	182	191	125	74	108	44
8	447	468	307	182	265	108
9	1097	1148	754	447	650	265
10	2692	2817	1851	1097	1595	650

Adding the values in our final row, we get that in total there are $a_{10} + b_{10} + c_{10} + d_{10} + e_{10} + f_{10} = 2692 + 2817 + 1851 + 1097 + 1595 + 650 = \boxed{10702}$ ways to arrange the eggs.

Congratulations to **Carol Lu** for submitting the correct answer!

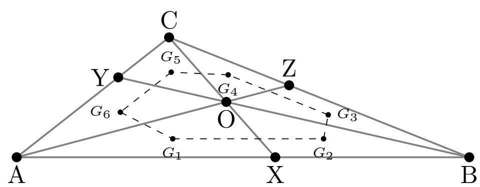
30 Problem of the Week 4/8/18 - 4/14/18

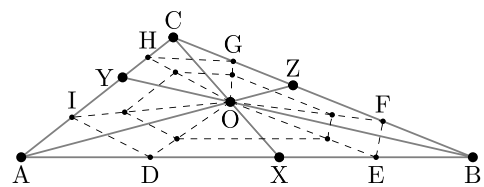
Consider $\triangle ABC$ with $\overline{AB} = 7$, $\overline{BC} = 5$, and $\overline{AC} = 3$. Consider points X, Y, and Z on segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively, such that $\frac{\overline{XB}}{\overline{XA}} = \frac{3}{4}$, $\frac{\overline{YC}}{\overline{YA}} = \frac{1}{2}$, and $\frac{\overline{ZC}}{\overline{ZB}} = \frac{2}{3}$. Let the intersection of $\overline{CX}$, $\overline{BY}$, and $\overline{AZ}$ be O.

Define the *centroid* of a triangle $\triangle PQR$ to be the intersection of the *medians* of $\triangle PQR$. The *median* of vertex P of $\triangle PQR$ is the segment between P and the midpoint of $\overline{QR}$, and the *medians* of Q and R are defined similarly.

Let the *centroids* of triangles $\triangle AXO$, $\triangle OXB$, $\triangle BOZ$, $\triangle OZC$, $\triangle OCY$, and $\triangle OYA$ be G_1, G_2, G_3, G_4, G_5, and G_6, respectively.

Given that the area of $\triangle ABC$ is $\dfrac{15\sqrt{3}}{4}$, compute the area of hexagon $G_1G_2G_3G_4G_5G_6$.





Solution: Let the midpoint of $\overline{AX}$ be D, let the midpoint of $\overline{XB}$ be E, let the midpoint of $\overline{BZ}$ be F, let the midpoint of $\overline{ZC}$ be G, let the midpoint of $\overline{CY}$ be H, and let the midpoint of $\overline{YA}$ be I. We know that the centroid splits any median in a $2:1$ ratio, so it follows that $\frac{\overline{OG_1}}{\overline{OD}} = \frac{\overline{OG_2}}{\overline{OE}} = \frac{\overline{OG_3}}{\overline{OF}} = \frac{\overline{OG_4}}{\overline{OG}} = \frac{\overline{OG_5}}{\overline{OH}} = \frac{\overline{OG_6}}{\overline{OI}} = \frac{2}{3}$. It follows that there is a homothety centered at O which brings hexagon $G_1G_2G_3G_4G_5G_6$ to $DEFGHI$ with a $\frac{3}{2}$ scale factor. It follows that we wish to calculate $(\frac{2}{3})^2 = \frac{4}{9}$ of the area of hexagon $DEFGHI$.

The area of hexagon $DEFGHI$ can be thought of as the difference between the area of $\triangle ABC$ and the sum of the areas of triangles $\triangle CHG$, $\triangle IAD$, and $\triangle BEF$. By the fact that the area of a triangle is $\frac{ab\sin C}{2}$, we know that the ratio of the area of $\triangle CHG$ to the area of $\triangle ABC$ is $\frac{0.5\cdot 1}{3\cdot 5} = \frac{1}{30}$. Similarly, the ratio of the area of $\triangle AID$ to the area of $\triangle ABC$ is $\frac{1\cdot 2}{3\cdot 7} = \frac{2}{21}$ and the ratio of the area of $\triangle EFB$ to the area of $\triangle ABC$ is $\frac{1.5\cdot 1.5}{5\cdot 7} = \frac{9}{140}$. It follows that the ratio of the sum of these areas to the area of $\triangle ABC$ is $\frac{9}{140} + \frac{1}{30} + \frac{2}{21} = \frac{81}{420} = \frac{27}{140}$. Therefore, our answer is $\frac{4}{9} \cdot (1 - \frac{27}{140}) \cdot \frac{15\sqrt{3}}{4} = \boxed{\dfrac{113\sqrt{3}}{84}}$.

Congratulations to **Carol Lu** and **Nathan Kuo** for submitting the correct answer!

31 Problem of the Week 4/15/18 - 4/21/18

Two nonzero real numbers a and b satisfy the equations

$$\frac{1}{a^5} + \frac{15}{a^3b^2} + \frac{6}{b^5} = 420a \text{ and } \frac{1}{b^5} + \frac{6}{a^5} + \frac{20}{a^3b^2} + \frac{15}{a^2b^3} = 309b$$

Given that $\frac{1}{a} + \frac{1}{b}$ is a positive real number, compute $207 \cdot \dfrac{ab}{a+b}$.

Solution: Dividing the first equation by a and the second equation by b gives us:

$$\frac{1}{a^6} + \frac{15}{a^4b^2} + \frac{6}{ab^5} = 420 \text{ and } \frac{6}{a^5b} + \frac{20}{a^3b^3} + \frac{15}{a^2b^4} + \frac{1}{b^6} = 309$$

Adding these two equations gives us:

$$\frac{1}{a^6} + \frac{6}{a^5b} + \frac{15}{a^4b^2} + \frac{20}{a^3b^3} + \frac{15}{a^2b^4} + \frac{6}{ab^5} + \frac{1}{b^6} = \left(\frac{1}{a} + \frac{1}{b}\right)^6 = 729$$

It follows that $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab} = 3$, and therefore $\dfrac{ab}{a+b} = \dfrac{1}{3}$. It follows that our answer is $207 \cdot \dfrac{1}{3} = \boxed{69}$.

Congratulations to **Carol Lu** and **Brycklen Arnold** for submitting the correct answer!

32 Problem of the Week 4/22/18 - 4/28/18

Consider the cubic $f(x) = x^3 - 640x^2 + 857x - 217$. Let the roots of $f(x)$ be a_1, a_2, and a_3, where each of a_1, a_2, and a_3 may or may not be real. Compute the sum:

$$S = \left| \frac{1}{1-a_1} + \frac{1}{1-a_2} + \frac{1}{1-a_3} \right|$$

Solution: Let $a_1 + a_2 + a_3 = d$, and let $a_1a_2 + a_1a_3 + a_2a_3 = e$, and let $a_1a_2a_3 = f$. We can notice that $(1 - a_1) + (1 - a_2) + (1 - a_3) = 3 - (a_1 + a_2 + a_3) = 3 - d$. Additionally, we can notice that $(1-a_1)(1-a_2)+(1-a_1)(1-a_3)+(1-a_2)(1-a_3) = 3-2(a_1+a_2+a_3)+a_1a_2+a_1a_3+a_2a_3 = 3-2d+e$. Finally, we can notice that $(1-a_1)(1-a_2)(1-a_3) = 1-(a_1+a_2+a_3)+(a_1a_2+a_1a_3+a_2a_3)-a_1a_2a_3 = 1 - d + e - f$. It follows that $1 - a_1$, $1 - a_2$, and $1 - a_3$ are the roots of the cubic

$$g(x) = x^3 - (3 - d)x^2 + (3 - 2d + e)x - (1 + e - d - f)$$

It follows that we wish to calculate the sum of the reciprocals of the roots of $g(x)$. By Vieta's Formulas, we know that $d = 640$, $e = 857$, and $f = 217$. It follows that we can write $g(x)$ as

$$g(x) = x^3 + 637x^2 - 420x - 1$$

It is well known that the sum of the reciprocals of the roots of a polynomial $h(x) = c_nx^n + \cdots + c_1x^1 + c_0$ is $-\frac{c_1}{c_0}$. It follows that the sum of the reciprocals is

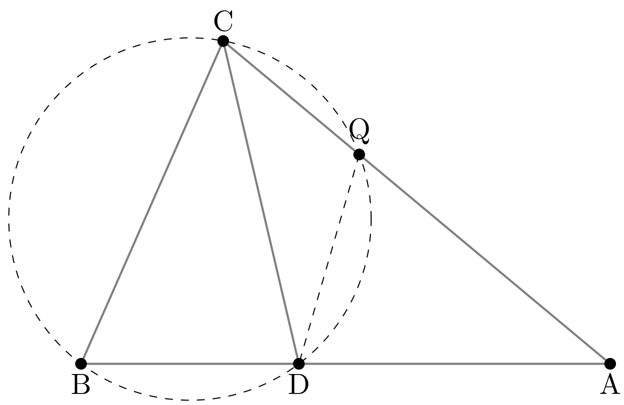
$$\frac{1}{1-a_1} + \frac{1}{1-a_2} + \frac{1}{1-a_3} = -\frac{-420}{-1} = -420$$

It follows that $S = |-420| = \boxed{420}$ as desired.

Congratulations to **Carol Lu** for submitting the correct answer!

33 Problem of the Week 4/29/18 - 5/5/18

Consider $\triangle ABC$ where $\overline{AB} = 21$, $\overline{AC} = 20$, and $\overline{BC} = 14$. Let D be the intersection of the angle bisector of $\angle ACB$ with $\overline{AB}$. Let Q be the point on line $\overline{AC}$ besides C that lies on the circumcircle of triangle $\triangle CBD$. That is, let Q be the point on $\overline{AC}$ such that points C, B, D, and Q all lie on one circle. Let the perimeter of quadrilateral $CBDQ$ be $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m + n$.



Solution: By the Angle Bisector Theorem, we know that $\dfrac{\overline{BD}}{\overline{AD}} = \dfrac{\overline{BC}}{\overline{AC}}$. It follows that $\overline{BD} = \frac{14}{14+20} \cdot 21 = \frac{147}{17}$. Because $\angle BCD \cong \angle DCQ$, we know that arc $\overset{\frown}{BD} \cong \overset{\frown}{QD}$. It follows that $\overline{QD} = \overline{BD} = \frac{147}{17}$. Because quadrilateral $BCQD$ is cyclic, we know that $\angle BDQ = 180° - \angle BCQ$. It follows that $\angle BCA \cong \angle ADQ$. Because $\angle DAQ \cong \angle CAB$, it follows that triangle $\triangle DAQ$ is similar to triangle $\triangle CAB$. Because $\dfrac{\overline{DQ}}{\overline{CB}} = \frac{21}{34}$, it follows that $\overline{AQ} = \frac{21}{34} \cdot 21 = \frac{441}{34}$. It follows that $\overline{CQ} = 20 - \frac{441}{34} = \frac{239}{34}$. Therefore, the perimeter of quadrilateral $CBDQ$ is $14 + \frac{147}{17} + \frac{147}{17} + \frac{239}{34} = \frac{1303}{34}$. It follows that our answer is $1303 + 34 = \boxed{1337}$.

Congratulations to **Carol Lu** and **Brycklen Arnold** for submitting the correct answer!

34 Problem of the Week 5/6/18 - 5/12/18

Define a n-digit binary sequence to be a sequence of 1's and 0's that has n total digits. For example, 000, 110, and 010 are all examples of 3-digit binary sequences. Define a *run* in the n-digit binary sequence to be a subsequence of consecutive digits in the sequence where all digits in the *run* are the same and it is impossible to extend the *run* in either direction. For example, in the 12-digit binary sequence 110001000011, the *runs* going from left to right are 11, 000, 1, 0000, and 11.

Define the *value* of an n-digit binary sequence to be the product of the lengths of each of the *runs* in the n-digit binary sequence. For example, the *value* of the 12-digit binary sequence 110001000011 is $2 \cdot 3 \cdot 1 \cdot 4 \cdot 2 = 48$.

Among all of the different possible 12-digit binary sequences, what is the average *value* of such a sequence?

Solution: Let A_n be the average *value* of a binary sequence with n digits. Notice that the average *value* of a binary sequence which ends in 1 is the same as the average *value* of a binary sequence of the same length which ends in 0. It follows that we can ignore what kind of digit is at the end of our binary sequence and instead worry about the length of the final *run* in the sequence. The probability that the final *run* in the sequence has k digits where $k < n$ is $\frac{1}{2^{k-1}} \cdot \frac{1}{2} = \frac{1}{2^k}$, as the probability that digits $n-k+1$ through $n-1$ are all the same as digit n is $\frac{1}{2^{k-1}}$ and the probability that digit $n-k$ is not the same is $\frac{1}{2}$. The probability that the final *run* in the sequence has n digits is $\frac{1}{2^{n-1}}$. It follows that we can recursively define A_n as the following:

$$A_n = \sum_{x=1}^{n-1} \left(A_x \cdot (n-x) \cdot \frac{1}{2^{n-x}} \right) + A_0 \cdot 1 \cdot \frac{1}{2^{n-1}}$$

Using this recursion, we can generate the following sequence: $A_0 = 1$, $A_1 = 1$, $A_2 = \frac{3}{2}$, $A_3 = 2$, $A_4 = \frac{21}{8}$, $A_5 = \frac{55}{16}$, $A_6 = \frac{9}{2}$, $A_7 = \frac{377}{64}$, $A_8 = \frac{987}{128}$, $A_9 = \frac{323}{32}$, $A_{10} = \frac{6765}{512}$, $A_{11} = \frac{17711}{1024}$, and finally, $A_{12} = \boxed{\frac{1449}{64}}$ as desired.

Nobody solved this problem of the week.

35 Problem of the Week 5/13/18 - 5/19/18

Let a, b, c, x, y, and z be real numbers which simultaneously satisfy the equations:

$$69x + by + cz = 0$$

$$ax + 420y + cz = 0$$

$$ax + by + 1337z = 0$$

Given that $a \neq 69$ and $x \neq 0$, compute the sum:

$$S = \frac{a}{a-69} + \frac{b}{b-420} + \frac{c}{c-1337}$$

Solution: Denote the three equations as $(1), (2)$, and (3) for future reference. Subtracting (1) from (2), we can find that $x(a-69) = y(b-420)$. Similarly, if we subtract (1) from (3), we can find that $x(a-69) = z(c-1337)$. It follows that $x(a-69) = y(b-420) = z(c-1337)$. Subtracting $y(b-420)+z(c-1337) = 2x(a-69)$ from (1) gives us $69x+420y+1337z = 2x(69-a)$. Similarly, we have that $69x + 420y + 1337z = 2x(69-a) = 2y(420-b) = 2z(1337-c)$. It follows that $\frac{1}{a-69} = \frac{-2x}{69x+420y+1337z}$, $\frac{1}{b-420} = \frac{-2y}{69x+420y+1337z}$, and $\frac{1}{c-1337} = \frac{-2z}{69x+420y+1337z}$. It follows that $\frac{69}{a-69} + \frac{420}{b-420} + \frac{1337}{c-1337} = \frac{-138x-840y-2674z}{69x+420y+1337z} = -2$. Notice that if we add 1 to each of the fractions in the sum, we get the sum in S, and it follows that our answer is $-2+1+1+1 = \boxed{1}$.

Congratulations to **Carol Lu** for submitting the correct answer!

36 End of the Year

Thanks for participating in the 2017 − 2018 UHS Math Club Problem of the Week!

Have a great summer, and see you next year!

37 Problem of the Week Leaderboard

Problem of the Week Leaderboard 2017-2018

Rank	Name	Problems Solved
1	Carol Lu	19
2	Tingting Thompson	6
3	Christopher Lehman	3
3	Brycklen Arnold	3
5	Joaquin Pesqueira	2
5	Orlando Rodriguez	2
5	Ernesto Alcala	2
8	Colin Hannan	1
8	Gabe Mogollon	1
8	Sam Roth Gordon	1
8	Ellie Standifer	1
8	Nickolas Cosentino	1
8	Grace Driskill	1
8	Maxwell Thum	1
8	Nathan Kuo	1