Number Theory Handout #6 Answers and Solutions

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1 Answers

1. 3

2. 495

3. 4028

4. 150

5. 720

6. 16

7. $\frac{21}{4}$

8. $2\ln 2 - 1$

9. 125

10. 9

11. 7

12. 650

13. 51

14. 605

15. 3640

2 Solutions

1. Let $S(n)$ denote the sum of the digits of the integer n. If $S(n) = 2018$, what is the smallest possible value $S(n+1)$ can be?

Solution: Recalling that the sum of the digits of n $S(n) \equiv n \pmod 9$, we know that $S(n+1) \equiv S(n)+1 \equiv 2019 \equiv 3 \pmod 9$. Therefore, $S(n+1) \geq 3$. This can be achieved when $n = 3*10^{224}-1$. Therefore, the smallest possible value of $S(n+1)$ is $\boxed{3}$.

2. One of the six digits in the expression $435 \cdot 605$ can be changed so that the product is a perfect square N^2. Compute N.

Solution: Notice that exactly one of the terms in this expansion is a factor of N^2. $435 = 3^1 \cdot 5^1 \cdot 29^1$. $605 = 5^1 \cdot 11^2$. Clearly 435 must be the term we need to change, as otherwise we would need the other term to be 435 multiplied by a perfect square, which is impossible. Therefore, we need to change a digit of 435 so we get 5^1 multipied by a perfect square. This can easily be achieved if we change 435 to $405 = 5^1 \cdot 3^4$. Therefore, $N = \sqrt{5^1 \cdot 11^2 \cdot 5^1 \cdot 3^4} = 3^2 \cdot 5^1 \cdot 11^1 = \boxed{495}$.

3. A sequence is defined as follows. Given a term a_n, we define the next term a_{n+1} as

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ a_n - 1 & \text{if } a_n \text{ is odd} \end{cases}$$

The sequence terminates when $a_n = 1$. Let $P(x)$ be the number of terms in such a sequence with initial term x. For example, $P(7) = 5$ because its corresponding sequence is $7, 6, 3, 2, 1$. Evaluate $P(2^{2018} - 2018)$.

Solution: Through brute force, we can find that if $a_1 = 2^{2018} - 2018$, then $a_{16} = 2^{2007} - 1$. Now from here we can notice that if $a_{2n} = 2^x - 1$, then $a_{2n+1} = 2^x - 2$ and $a_{2n+2} = 2^{x-1} - 1$. Therefore for $n \geq 8$, $a_{2n} = 2^{2015-n} - 1$ for many terms. It follows that $a_{4026} = 2^2 - 1 = 3$. Therefore, $a_{4027} = 2$ and $a_{4028} = 1$. It follows that our answer is $\boxed{4028}$.

4. Elizabeth is at a candy store buying jelly beans. Elizabeth begins with 0 jellybeans. With each scoop, she can increase her jellybean count to the next largest multiple of $30, 70$, or 110. (For example, her next scoop after 70 can increase her jellybean count to $90, 110$, or 140). What is the smallest number of jellybeans Elizabeth can collect in more than 100 different ways?

Solution: Notice that the answer has to be a multiple of $30, 70$, or 110. Let the answer be a, and let the number of positive multiples of $30, 70$, or 110 that are less than or equal to a be n. Notice that the total number of ways to collect a jelly beans is 2^{n-1}, as each of the first $n-1$ multiples can either be "visited" or not "visited", where we define a multiple to be "visited" if and only if it is the total number of jelly beans collected after some scoop in the collection process. It follows that we want to find the 8th positive integer which is a multiple of at least one of $30, 70$, or 110. Listing these numbers, we get the list $30, 60, 70, 90, 110, 120, 140, 150$. Therefore, our answer is $\boxed{150}$.

5. Positive integer n can be written in the form $a^2 - b^2$ for at least 12 pairs of positive integers (a, b). Compute the smallest possible value of n.

Solution: Notice that $n = (a-b)(a+b)$ and $a - b$ and $a + b$ have the same parity. Therefore, either n is divisible by 4 or n is odd. If n were divisible by 4, then if n had x even divisors of the form $2a$ such that $\frac{n}{2a}$ is an even integer, then there would be $\lceil \frac{x}{2} \rceil$ pairs (a, b) which would work. However, if n is odd, then if n has x odd factors, there would be $\lfloor \frac{x}{2} \rfloor$ pairs (a, b) which would work. Clearly we can make n smaller by making n even, as powers of 2 grow much less quickly than other prime powers. If we let $n = 2^a \cdot b$ where b has x factors, then the number of pairs which work is $\lfloor \frac{ax-x}{2} \rfloor$. Therefore, we wish to find the smallest value of n of this form such that $ax - x \geq 24$. Because 24 has lots of small factors, it seems n will be minimized when $ax - x = 24$. If $a = 7$, then we can let $b = 15$ for the value $n = 2^7 \cdot 3^1 \cdot 5^1$. If $a = 5$, then we can let $b = 45$ for the value $n = 2^4 \cdot 3^2 \cdot 5^1$. Through brute force, we can find that this value does minimize n, and therefore our answer is $2^4 \cdot 3^2 \cdot 5^1 = \boxed{720}$.

6. Let

$$S = \sum_{k=1}^{2018102} \sum_{n=1}^{1008} n^k.$$

Compute the remainder when S is divided by 1009.

Solution: We will begin by proving that whenever p is prime and k is a positive integer, we have that

$$\sum_{n=1}^{p-1} n^k \equiv p - 1 \pmod{p}$$

when $p-1 \mid k$, and otherwise we have

$$\sum_{n=1}^{p-1} n^k \equiv 0 \pmod{p}.$$

First consider if $p-1 \mid k$. By Fermat's Little Theorem, we know that $n^{p-1} \equiv 1 \pmod{p}$, and it follows that $n^k \equiv 1 \pmod{p}$. Therefore, the sum is

$$\sum_{n=1}^{p-1} n^k \equiv \sum_{n=1}^{p-1} 1 \equiv p-1 \pmod{p}$$

Now consider when $p-1 \nmid k$. It is well known that when p is prime, there exists a primitive root $r \pmod{p}$ such that $r^{p-1} \equiv 1 \pmod{p}$ and for $1 \leq n \leq p-1$, $r^n \not\equiv 1 \pmod{p}$. It follows by the Pigeonhole Principle that for $1 \leq n \leq p-1$, there exists exactly one integer $0 \leq i < p-1$ such that $r^i \equiv n \pmod{p}$. It follows that we can rewrite this sum as

$$\sum_{n=1}^{p-1} n^k \equiv \sum_{i=1}^{p-1} r^{ki} \pmod{p}$$

Now let this simplifed version of the sum be O. Then we know that

$$O\cdot(1-r^k) \equiv \sum_{i=1}^{p-1} r^{ki} - r^{ki+k} \equiv r^k - r^{2k} + r^{2k} - r^{3k} + \cdots + r^{pk-k} - p^k \equiv r^k - r^{pk} \equiv r^k\cdot(1-r^{(p-1)k}) \equiv r^k\cdot(1-1^k) \equiv 0 \quad (\text{mod}$$

In other words, $O \cdot (1-r^k) \equiv 0 \pmod{p}$. By definition, because $p-1 \nmid k$, we know $1-r^k \not\equiv 0 \pmod{p}$. It follows that $O \equiv 0 \pmod{p}$, and therefore, we must have

$$O \equiv \sum_{n=1}^{p-1} n^k \equiv 0 \pmod{p}$$

It follows because $p = 1009$ is prime that the given sum is equivalent to

$$S \equiv \sum_{k=1}^{\lfloor \frac{2018102}{1008} \rfloor} \sum_{n=1}^{1008k} n^k \equiv -\lfloor \frac{2018102}{1008} \rfloor \equiv \boxed{16} \pmod{1009}$$

7. Let $f(k)$ be a function defined by the following rules:

(a) $f(k)$ is multiplicative. In other words, if $\gcd(a,b) = 1$, then $f(ab) = f(a) \cdot f(b)$,
(b) $f(p^k) = k$ for primes $p = 2, 3$ and all $k > 0$,
(c) $f(p^k) = 0$ for primes $p > 3$ and all $k > 0$, and
(d) $f(1) = 1$.

For example, $f(12) = 2$ and $f(160) = 0$. Evaluate

$$\sum_{k=1}^{\infty} \frac{f(k)}{k}.$$

Solution: Under the given conditions, we know that $f(2^a \cdot 3^b) = ab$ for positive integers a and b, $f(2^a) = a$ for positive integers a, $f(3^b) = b$ for positive integers b, $f(1) = 1$, and $f(n) = 0$ for all numbers n not of one of the above forms. Under these conditions, we can find that we wish to calculate

$$\sum_{k=1}^{\infty} \frac{f(k)}{k} = 1 + \sum_{a=1}^{\infty} \frac{a}{2^a} + \sum_{b=1}^{\infty} \frac{b}{3^b} + \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{ab}{2^a \cdot 3^b}.$$

Notice that $\sum_{a=1}^{\infty} \frac{a}{2^a} = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) + (\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots) + (\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots) + \cdots$. By the formula for an infinite geometric series, we know this is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. Using similar reasoning we can find that $\sum_{b=1}^{\infty} \frac{b}{3^b} = \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \cdots = \frac{3}{4}$. Now for the fourth term in our total sum, we can rewrite the sum as

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{ab}{2^a \cdot 3^b} = \sum_{a=1}^{\infty} \frac{a}{2^a} \cdot \sum_{b=1}^{\infty} \frac{b}{3^b} = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

Therefore our answer is $1 + 2 + \frac{3}{4} + \frac{3}{2} = \boxed{\dfrac{21}{4}}$.

8. Consider all increasing arithmetic progressions of the form $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ such that $a, b, c \in \mathbb{N}$ and $\gcd(a, b, c) = 1$. Find the sum of all possible values of $\frac{1}{a}$.

Solution: Let the common difference of this arithmetic series be d for some rational number d. It follows that we can rewrite this sequence as $\frac{1}{b} = \frac{1+da}{a}$ and $\frac{1}{c} = \frac{1+2da}{a}$. Because we need the simplified version of $\frac{1+da}{a}$ to have a numerator of 1, we must have that $1 + da \mid a$, or in other words, we must have that $\frac{a}{1+da}$ is an integer. Similarly, we must have that $\frac{a}{1+2da}$ is an integer. It follows that we can write $a = r(1 + da)(1 + 2da)$ for some rational number r. However, if r were not equal to 1, then we would either have that $\gcd(a, b, c) \neq 1$ or we would have that at least one of a, b, and c would not be an integer. Therefore, we must have $a = (1 + da)(1 + 2da)$. From this it follows that da is an integer because we can rewrite $\frac{1}{c} = \frac{1+2da}{a} = \frac{1}{1+da}$. Therefore, if we let $da = y$, then we know that $a = (1 + y)(1 + 2y)$ for some integer y. It follows that the general solution for this type of arithmetic progression is the sequence

$$\frac{1}{(1+y)(1+2y)}, \frac{1}{1+2y}, \frac{1}{1+y}$$

for some integer $y \geq 1$. We can easily check that this will work for any integer $y \geq 1$, so it follows that we wish to compute

$$\sum_{y=1}^{\infty} \frac{1}{(1+y)(1+2y)} = \sum_{y=1}^{\infty} \frac{2}{1+2y} - \frac{1}{1+y} = \sum_{y=1}^{\infty} \frac{2}{1+2y} - \frac{2}{2+2y}$$

This sum rearranges to $2 \cdot (\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots)$. It is well known that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, so it follows that our answer is $2 \cdot (\ln 2 - 1 + \frac{1}{2}) = \boxed{2 \ln 2 - 1}$

9. How many ways are there to select distinct integers x, y where $1 \leq x \leq 25$ and $1 \leq y \leq 25$, such that $x + y$ is divisible by 5?

Solution: Notice that regardless of what the value of x is, there are always exactly 5 values of y in this range which satisfy $x + y$ is divisible by 25. It follows that our answer is $25 \cdot 5 = \boxed{125}$.

10. How many integer pairs (a, b) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{2018}$?

Solution: Simplifying this equation, we get $ab = 2018a + 2018b$. Using Simon's Favorite Factoring Trick (SFFT), we can rearrange this as $(a - 2018)(b - 2018) = 2018^2$. From here it follows that any positive integer factor pair of 2018^2 corresponds to a unique pair (a, b) which satisfies the original equation. It follows that our answer is the number of factors of $2018^2 = 2^2 \cdot 1009^2$. It follows that our answer is $(2+1)^2 = \boxed{9}$.

11. Positive integer n has the property such that $n - 64$ is a positive perfect cube. Suppose that n is divisible by 37. What is the smallest possible value of n?

Solution: Let $n - 64 = x^3$. It follows that $x^3 + 4^3 = n$, or $(x+4)(x^2 - 4x + 16) = n$. The smallest value of x which satisfies $x + 4 \equiv 0 \pmod{37}$ is $x = 33$. The smallest value of x which satisfies $x^2 - 4x + 16 \equiv 0 \pmod{37}$ is $x = 7$, as x^3 must be positive. Therefore our answer is $\boxed{7}$.

12. Stu is on a train en route to SMT. He is bored, so he starts doodling in his notebook. Stu realizes that he can combine SMT as an alphametic, where each letter represents a unique integer and the leading digits may not be zero, to get his name as shown: $\sqrt{SMT} + SMT = STU$. Find the three digit number STU.

Solution: Notice that by the given equation, we must have SMT is a 3-digit perfect square. Also, due to the fact that $\sqrt{SMT} < \sqrt{1000}$, or $\sqrt{SMT} \leq 31$, we know that $T \leq M + 4$. Using this knowledge, we can easily brute force this question by looking at all 3-digit perfect squares which satisfy this property. Doing so gives us that if $SMT = 625$, then $625 + \sqrt{625} = 650$, and therefore all properties are satisfied. It follows that our answer is $\boxed{650}$.

13. A 3×3 magic square is a grid of **distinct** numbers whose rows, columns, and diagonals all add to the same integer sum. Connie creates a magic square whose sum is N, but her keyboard is broken so that when she types a number, one of the digits $(0-9)$ always appears as a different digit (e.g. if the digit 8 always appears as 5, the number 18 will appear as 15). The altered square is shown below. Find N.

9	11	10
18	17	6
14	11	15

Solution: Note that the row, column, and diagonal totals of the given square are $30, 41, 40, 41, 39, 31, 41$, and 41. Given the wide variety between a few of these totals, it is safe to say that at least some of the tens-digits are not actually 1s. Based on the small difference in the units digits of these totals, it makes sense that the digit that is being changed is 1 away from the digit that it is changed to. As a result, we can easily assume that her keyboard changes a 2 to a 1. From here, we can find through brute force that the following magic square could have produced this square:

9	22	20
28	17	6
14	12	25

As a result, our answer is $9 + 22 + 20 = \boxed{51}$.

14. Positive integer n has 6 factors including n and 1. Suppose that the third largest factor of n, including n, is 55. Compute n.

Solution: Notice that if n has 6 factors, then either $n = p^5$ for some prime p or $n = p^2 \cdot q^1$ for two distinct primes p and q. Because $55 = 5^1 \cdot 11^1$ is a factor of n, we know that $n = p^2 \cdot q^1$ where (p, q) is a rearrangement of $(5, 11)$. Testing, we find that 55 is the third smallest factor of $5^1 \cdot 11^2 = \boxed{605}$.

15. How many 5 digit numbers n exist such that each n is divisible by 9 and *none* of the digits of n are divisible by 9?

Solution: We wish to find the number of solutions to $a+b+c+d+e \equiv 0 \pmod 9$ where each of a, b, c, d, e is among the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Notice that for every choice of the digits a, b, c, d such that $a+b+c+d$ is not divisible by 9, there is exactly one value of e which will make the 5-tuple satisfy this property. Therefore, if we let F_4 be the number of 4-digit numbers with this property, then our answer is $8^4 - F_4$. In general, if we let F_n be the number of n-digit numbers with this property, then for $n \geq 2$, we have $F_n = 8^{n-1} - F_{n-1}$. We can easily find that $F_1 = 0$, and using our recursion, we can find that $F_2 = 8$, $F_3 = 56$, $F_4 = 456$, and $F_5 = 8^4 - 456 = 4096 - 456 = \boxed{3640}$.

3 Sources

1. 2018 Stanford Math Tournament Discrete Problem 2
2. 2018 Stanford Math Tournament Discrete Problem 4
3. 2018 Stanford Math Tournament Discrete Problem 5
4. 2018 Stanford Math Tournament Discrete Problem 6
5. 2018 Stanford Math Tournament Discrete Problem 8
6. 2018 Stanford Math Tournament Discrete Problem 9
7. 2018 Stanford Math Tournament Team Problem 11
8. 2018 Stanford Math Tournament Team Problem 12
9. 2018 Stanford Math Tournament General Problem 15
10. 2018 Stanford Math Tournament General Problem 18
11. 2018 Stanford Math Tournament General Problem 21
12. 2018 Stanford Math Tournament General Problem 24
13. 2018 Stanford Math Tournament General Problem 19
14. 2018 Stanford Math Tournament Discrete Tiebreaker Problem 1
15. 2018 Stanford Math Tournament Discrete Tiebreaker Problem 2