Number Theory Handout #6

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1 Problems

1. Let $S(n)$ denote the sum of the digits of the integer n. If $S(n) = 2018$, what is the smallest possible value $S(n+1)$ can be?

2. One of the six digits in the expression $435 \cdot 605$ can be changed so that the product is a perfect square N^2. Compute N.

3. A sequence is defined as follows. Given a term a_n, we define the next term a_{n+1} as

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ a_n - 1 & \text{if } a_n \text{ is odd} \end{cases}$$

The sequence terminates when $a_n = 1$. Let $P(x)$ be the number of terms in such a sequence with initial term x. For example, $P(7) = 5$ because its corresponding sequence is $7, 6, 3, 2, 1$. Evaluate $P(2^{2018} - 2018)$.

4. Elizabeth is at a candy store buying jelly beans. Elizabeth begins with 0 jellybeans. With each scoop, she can increase her jellybean count to the next largest multiple of $30, 70$, or 110. (For example, her next scoop after 70 can increase her jellybean count to $90, 110$, or 140). What is the smallest number of jellybeans Elizabeth can collect in more than 100 different ways?

5. Positive integer n can be written in the form $a^2 - b^2$ for at least 12 pairs of positive integers (a, b). Compute the smallest possible value of n.

6. Let

$$S = \sum_{k=1}^{2018102} \sum_{n=1}^{1008} n^k.$$

Compute the remainder when S is divided by 1009.

7. Let $f(k)$ be a function defined by the following rules:

(a) $f(k)$ is multiplicative. In other words, if $\gcd(a, b) = 1$, then $f(ab) = f(a) \cdot f(b)$,
(b) $f(p^k) = k$ for primes $p = 2, 3$ and all $k > 0$,
(c) $f(p^k) = 0$ for primes $p > 3$ and all $k > 0$, and
(d) $f(1) = 1$.

For example, $f(12) = 2$ and $f(160) = 0$. Evaluate

$$\sum_{k=1}^{\infty} \frac{f(k)}{k}.$$

8. Consider all increasing arithmetic progressions of the form $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ such that $a, b, c \in \mathbb{N}$ and $\gcd(a, b, c) = 1$. Find the sum of all possible values of $\frac{1}{a}$.

9. How many ways are there to select distinct integers x, y where $1 \leq x \leq 25$ and $1 \leq y \leq 25$, such that $x + y$ is divisible by 5?

10. How many integer pairs (a, b) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{2018}$?

11. Positive integer n has the property such that $n - 64$ is a positive perfect cube. Suppose that n is divisible by 37. What is the smallest possible value of n?

12. Stu is on a train en route to SMT. He is bored, so he starts doodling in his notebook. Stu realizes that he can combine SMT as an alphametic, where each letter represents a unique integer and the leading digits may not be zero, to get his name as shown: $\sqrt{SMT} + SMT = STU$. Find the three digit number STU.

13. A 3×3 magic square is a grid of **distinct** numbers whose rows, columns, and diagonals all add to the same integer sum. Connie creates a magic square whose sum is N, but her keyboard is broken so that when she types a number, one of the digits $(0 - 9)$ always appears as a different digit (e.g. if the digit 8 always appears as 5, the number 18 will appear as 15). The altered square is shown below. Find N.

9	11	10
18	17	6
14	11	15

14. Positive integer n has 6 factors including n and 1. Suppose that the third largest factor of n, including n, is 55. Compute n.

15. How many 5 digit numbers n exist such that each n is divisible by 9 and *none* of the digits of n are divisible by 9?

2 Sources

1. 2018 Stanford Math Tournament Discrete Problem 2
2. 2018 Stanford Math Tournament Discrete Problem 4
3. 2018 Stanford Math Tournament Discrete Problem 5
4. 2018 Stanford Math Tournament Discrete Problem 6
5. 2018 Stanford Math Tournament Discrete Problem 8
6. 2018 Stanford Math Tournament Discrete Problem 9
7. 2018 Stanford Math Tournament Team Problem 11
8. 2018 Stanford Math Tournament Team Problem 12
9. 2018 Stanford Math Tournament General Problem 15
10. 2018 Stanford Math Tournament General Problem 18
11. 2018 Stanford Math Tournament General Problem 21
12. 2018 Stanford Math Tournament General Problem 24
13. 2018 Stanford Math Tournament General Problem 19
14. 2018 Stanford Math Tournament Discrete Tiebreaker Problem 1
15. 2018 Stanford Math Tournament Discrete Tiebreaker Problem 2