Number Theory Handout # 7 Answers and Solutions

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1 Answers

1. 112

2. 371

3. 6561

4. 96

5. 7

6. 340

7. $\frac{1}{19}$

8. 9

9. 6

10. 7

11. 297

12. $27,720$

13. 325

14. 15

15. 0.21

2 Solutions

1. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Solution: We can find through brute force that the first few terms of this sequence are $2, 5, 8, 5, 6, 10, 7, 4, 7, 6, 2, 5, 8, \cdots$. It follows that the sequence repeats every 10 terms. It follows that $a_{2018} = 7$, $a_{2020} = 2$, and $a_{2022} = 8$. Therefore our answer is $7 \cdot 2 \cdot 8 = \boxed{112}$.

2. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

Solution: We want $3b + 6$ to be a perfect square and $2b + 7$ to be a perfect cube. Notice that because $3b + 6$ is a perfect square which is divisible by 3, it must be divisible by 9. It follows that $b \equiv 1 \pmod 3$. It follows that $2b + 7 \equiv 0 \pmod 3$. It follows that $2b + 7$ must be divisible by 27, and therefore $b \equiv 10 \pmod{27}$. It follows that if we let $b = 27c + 10$, then $54c + 27 = y^3$, and therefore $2c+1$ is a perfect cube. We must also have that $27c+10 \leq 1000$, so it follows that $c \leq 36$. From here, the only two possible values of c are $c = 0$ and $c = 13$. These result in the values $b = 10$ and $b = 361$, which we can easily confirm to work. It follows that our answer is $10 + 361 = \boxed{371}$.

3. How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

Solution: Notice that if we let the given expression be N, then the value $N + 3^7 + 3^6 + \cdots + 3^1 + 3^0$ can be expressed with an 8-digit base-3 (trinary) representation. It follows that the total number of numbers which work is $3^8 = \boxed{6561}$.

4. How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

Solution: Notice that for each hundreds digit and units digit, there are exactly 3 possible tens digits which make the number divisible by 3. It follows that our answer is $4 \cdot 8 \cdot 3 = \boxed{96}$.

5. Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

Solution: By definition, the given conditions imply that $\frac{p}{q}$ is adjacent to $\frac{5}{9}$ and $\frac{4}{7}$ in some term of the Farey Sequence, and therefore $5q + 1 = 9p$ and $7p + 1 = 4q$. It follows that $\frac{5}{4}(7p + 1) + 1 = 9p$, and solving we get $p = 9$. Plugging this into one of the other equations gives us $q = 16$, and we can check that the fraction $\frac{9}{16}$ works. Therefore our answer is $16 - 9 = \boxed{7}$.

6. Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

Solution: By the given conditions, we must have that $n < 10000$, $2 \mid n$ and $323 \mid n$. We can find that $323 = 17^1 \cdot 19^1$. Because $n = 626a$ for some integer a and $n < 10000$, we must have that $a < 16$. If we assume the number x that came after 323 in the list was not divisible by 17 or 19, then it would follow that $x \mid 2a \to x \leq 2a < 32 < 323$. It follows that x is divisible by 17 or 19. The smallest multiple of one of these primes which is greater than 323 is $340 = 17 \cdot 20$. We can notice that if $a = 10$, then this condition would be met and that n would be 6260. It follows that our answer is $\boxed{340}$.

7. The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

Solution: By Legendre's Theorem, we know that the largest power of 2 which divides 21! has an exponent of $\lfloor \frac{21}{2} \rfloor + \lfloor \frac{21}{4} \rfloor + \lfloor \frac{21}{8} \rfloor + \lfloor \frac{21}{16} \rfloor = 10 + 5 + 2 + 1 = 18$. Notice that for every odd factor of 21!, there are 18 even factors which uniquely correspond to it which can be obtained by multiplying that odd factor by an element of the set $\{2^1, 2^2, \cdots 2^{18}\}$. It follows that the probability that a randomly chosen factor is odd is $\frac{1}{1+18} = \boxed{\frac{1}{19}}$.

8. Let $N = 123456789101112...4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

Solution: By The Chinese Remainder Theorem (CRT), we can determine $N \pmod{45}$ from $N \pmod{5}$ and $N \pmod{9}$. The remainder when N is divided by 5 is clearly 4. If we let $S(n)$ be the sum of the digits of N, then we know that $N \equiv S(N) \equiv S(1) + S(2) + S(3) + \cdots + S(44) \equiv 1 + 2 + 3 + \cdots + 44 \equiv \frac{44 \cdot 45}{2} \equiv 0 \pmod{9}$. Therefore, we know that $N \equiv 4 \pmod{5}$ and $N \equiv 0 \pmod{9}$. It follows that $N \equiv \boxed{9} \pmod{45}$.

9. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. What is the remainder when $S(n+1)$ is divided by 9?

Solution: By the divisibility rule for 9, we know that $S(n) \equiv n \pmod{9}$. It follows that $S(n+1) \equiv n + 1 \equiv S(n) + 1 \equiv 1275 \equiv S(1275) \equiv \boxed{6} \pmod{9}$ as desired.

10. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

Solution: Consider a number n such that there exists a sequence of n consecutive positive integers with a sum of 345. Because the sum of the first 26 positive integers is $\frac{26 \cdot 27}{2} = 351 > 345$, we must have that $n < 26$. Consider when n is odd. If such a sequence exists, then it must have a middle term m. It follows that the sum of this sequence is mn. It follows that any number $n < 25$ which is a factor of 345 will work. We can easily find that $345 = 5^1 \cdot 3^1 \cdot 23^1$. It follows that $n = 3$, $n = 5$, $n = 15$, and $n = 23$ all work. Now consider when n is even. If we let $n = 2x$, and we let the middle two terms be a and $a + 1$, then we must have that $x(2a + 1) = 345$. We must have that $2x = n < 26$, so it follows that any factor x of 345 which is less than 13 will correspond to a value of n which satisfies this property. We can quickly find that $x = 1$, $x = 3$, and $x = 5$ all work, and these values correspond to $n = 2$, $n = 6$, and $n = 10$, respectively. It follows that there are exactly $\boxed{7}$ sequences which satisfy this property.

11. For a certain positive integer n less than 1000, the decimal equivalent of $\dfrac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period of 6, and the decimal equivalent of $\dfrac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. What is n?

Solution: By definition, we must have that $\frac{\overline{abcdef}}{999{,}999} = \frac{1}{n}$. In addition, we must have that $\frac{\overline{wxyz}}{9999} = \frac{1}{n+6}$. It follows that n must be a factor of 999999 but not a factor of 999 or 99 and $n + 6$ must be a factor of 9999 but not a factor of 99. It follows that $n + 6$ must be divisible by 101. We can find that if $n = 297$, then n is a factor of 999999 and $n + 6 = 303$ is a factor of 9999. Therefore our answer is $\boxed{297}$.

12. There are exactly 77,000 ordered quadruplets (a, b, c, d) such that $\gcd(a, b, c, d) = 77$ and $\text{lcm}(a, b, c, d) = n$. What is the smallest possible value for n?

Solution: Consider the quadruplet $Q = (\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77})$. We know that $\gcd(Q) = 1$ and we know that $\text{lcm}(Q) = \frac{n}{77}$. We are given that there are exactly 77,000 ordered quadruplets of the form Q. Consider an arbitrary prime p and let the largest power of p which divides n be p^e. Notice that for each prime power of this form, the number of quadruplets which satisfies these properties increases by a factor of $(e + 1)^4 - e^4 - e^4 + (e - 1)^4$ as there are $(e + 1)^4$ ways to give each element of Q a prime power factor of p^x where $0 \le x \le e$, there are e^4 ways to give each element of Q a prime power factor of p^x where $1 \le x \le e$, there are e^4 ways to give each element of Q a prime power factor of p^x where $0 \le x \le e - 1$, and there are $(e - 1)^4$ ways to give each element of Q a prime power factor of p^x where $1 \le x \le e - 1$. If we let $X(e) = (e + 1)^4 - 2e^4 + (e - 1)^4$, then it follows that $X(1) = 14$, $X(2) = 50$, $X(3) = 110$, $X(4) = 194$, and $X(5) = 302$. From here, we can notice that if n is of the form $p_1^3 p_2^2 p_3^1$, then the total number of quadruplets will be $14 \cdot 50 \cdot 110 = 77000$.

Therefore our answer is $77^1 \cdot 2^3 \cdot 3^2 \cdot 5^1 = \boxed{27,720}$.

13. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

Solution: Let $n = k \cdot 11^a \cdot 5^b \cdot 2^c$ where k is not divisible by 2, 5, or 11 and k has $d+1$ factors. Then we must have that $(3a+2)(3b+2)(3c+2)(d+1) = 110$ factors where a, b, and c are natural numbers and d is a nonnegative integer. Because 110 only has 3 prime factors, we must have that $d = 0$ and therefore $k = 1$. It follows that $(3a+2)(3b+2)(3c+2) = 110$. It follows that (a, b, c) is a rearrangement of $(0, 1, 3)$. We know that $81n^4$ has $5(4a+1)(4b+1)(4c+1)$ factors. It follows that our answer is $5 \cdot 13 \cdot 5 \cdot 1 = \boxed{325}$.

14. How many ordered triples (x, y, z) of positive integers satisfy $\text{lcm}(x, y) = 72$, $\text{lcm}(x, z) = 600$, and $\text{lcm}(y, z) = 900$?

Solution: Let 2^a be the largest power of 2 which divides x and define 2^b and 2^c similarly for y and z, respectively. Then we must have that $\max(a, b) = 3$, $\max(a, c) = 3$, and $\max(b, c) = 2$. We must have that $a = 3$, as otherwise both b and c would be equal to 3 which is impossible. We must have that at least one of b and c is 2. It follows that there are $2 \cdot 2 + 1 = 5$ possible triples (a, b, c) which work. Let 3^d be the largest power of 3 which divides x and define 3^e and 3^f similarly for y and z, respectively. Then we must have that $\max(d, e) = 2$, $\max(d, f) = 1$, and $\max(e, f) = 2$. It follows that $e = 2$, and one of d and f is equal to 1. It follows that there are $2 \cdot 1 + 1 = 3$ triples (d, e, f) which work. Finally, we can notice that because $\text{lcm}(x, y) = 72$ is not divisible by 5, we must have that 25 divides z. It follows that our answer is $3 \cdot 5 \cdot 1 = \boxed{15}$.

15. Back in 1930, Tillie had to memorize her multiplication facts from 0×0 to 12×12. The multiplication table she was given had rows and columns labeled with the factors, and the products formed the body of the table. To the nearest hundredth, what fraction of the numbers in the body of the table are odd?

Solution: Notice that a product is only odd if all numbers in the product are odd. The probability that a row number is odd is $\frac{6}{13}$ and the probability that a column number is odd is $\frac{6}{13}$. It follows that the fraction of the numbers in the grid which are odd is $(\frac{6}{13})^2 = \frac{36}{169}$. If we round this to the nearest 100th, we get $\boxed{0.21}$.

3 Sources

1. 2018 AIME II Problem 2
2. 2018 AIME II Problem 3
3. 2018 AMC 12A Problem 13
4. 2018 AMC 12B Problem 15
5. 2018 AMC 12B Problem 17
6. 2018 AMC 12B Problem 19
7. 2017 AMC 12B Problem 16
8. 2017 AMC 12B Problem 19
9. 2017 AMC 12A Problem 18 (Adapted)
10. 2016 AMC 12B Problem 16
11. 2016 AMC 12B Problem 22 (Adapted)
12. 2016 AMC 12B Problem 24
13. 2016 AMC 12A Problem 18

14. 2016 AMC 12A Problem 22

15. 2015 AMC 12B Problem 6