Number Theory Handout # 7

Walker Kroubalkian

April 3, 2018

1 Problems

1. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

2. Find the sum of all positive integers $b < 1000$ such that the base-b integer 36_b is a perfect square and the base-b integer 27_b is a perfect cube.

3. How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

4. How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

5. Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

6. Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

7. The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

8. Let $N = 123456789101112...4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

9. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. What is the remainder when $S(n + 1)$ is divided by 9?

10. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

11. For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period of 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. What is n?

12. There are exactly $77,000$ ordered quadruplets (a, b, c, d) such that $\gcd(a, b, c, d) = 77$ and $\text{lcm}(a, b, c, d) = n$. What is the smallest possible value for n?

13. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

14. How many ordered triples (x, y, z) of positive integers satisfy $\text{lcm}(x, y) = 72$, $\text{lcm}(x, z) = 600$, and $\text{lcm}(y, z) = 900$?

15. Back in 1930, Tillie had to memorize her multiplication facts from 0×0 to 12×12. The multiplication table she was given had rows and columns labeled with the factors, and the products formed the body of the table. To the nearest hundredth, what fraction of the numbers in the body of the table are odd?

2 Sources

1. 2018 AIME II Problem 2
2. 2018 AIME II Problem 3
3. 2018 AMC 12A Problem 13
4. 2018 AMC 12B Problem 15
5. 2018 AMC 12B Problem 17
6. 2018 AMC 12B Problem 19
7. 2017 AMC 12B Problem 16
8. 2017 AMC 12B Problem 19
9. 2017 AMC 12A Problem 18 (Adapted)
10. 2016 AMC 12B Problem 16
11. 2016 AMC 12B Problem 22 (Adapted)
12. 2016 AMC 12B Problem 24
13. 2016 AMC 12A Problem 18
14. 2016 AMC 12A Problem 22
15. 2015 AMC 12B Problem 6