Number Theory Handout #4

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1 Problems

1. The **three-digit** prime number p is written in base 2 as p_2 and in base 5 as p_5, and the two representations share the same last 2 digits. If the ratio of the number of digits in p_2 to the number of digits in p_5 is 5 to 2, find all possible values of p.

2. Find the sum of all possible n such that n is a positive integer and there exist a, b, c real numbers such that for every integer m, the quantity $\frac{2013m^3+am^2+bm+c}{n}$ is an integer.

3. A number is between 500 and 1000 and has a remainder of 6 when divided by 25 and a remainder of 7 when divided by 9. Find the only odd number to satisfy these requirements.

4. Given $f_1 = 2x - 2$ and $k \geq 2$, define $f_k(x) = f_1(f_{k-1}(x))$ to be a real-valued function of x. Find the remainder when $f_{2013}(2012)$ is divided by the prime 2011.

5. Consider the roots of the polynomial $x^{2013} - 2^{2013} = 0$. Some of these roots also satisfy $x^k - 2^k = 0$, for some integer $k < 2013$. What is the product of this subset of roots?

6. Denote by $S(a, b)$ the set of integers k that can be represented as $k = a \cdot m + b \cdot n$, for some non-negative integers m and n. So, for example, $S(2, 4) = \{0, 2, 4, 6, ...\}$. Then, find the sum of all possible positive integer values of x such that $S(18, 32)$ is a subset of $S(3, x)$.

7. Let σ_n be a permutation of $\{1, ..., n\}$; that is, $\sigma_n(i)$ is a bijective function from $\{1, ..., n\}$ to itself. Define $f(\sigma)$ to be the number of times we need to apply σ to the identity in order to get the identity back. For example, f of the identity is just 1, and all other permutations have $f(\sigma) > 1$. What is the smallest n such that there exists a σ_n with $f(\sigma_n) = 2013$?

8. Given that $468751 = 5^8 + 5^7 + 1$ is a product of two primes, find both of them.

9. How many zeros does the product of the positive factors of 10000 (including 1 and 10000) have?

10. Find the ordered pair of positive integers (x, y) such that $144x - 89y = 1$ and x is minimal.

11. Let m and n be integers such that $m + n$ and $m - n$ are prime numbers less than 100. Find the maximal possible value of mn.

12. For a positive integer n, let $\phi(n)$ denote the number of positive integers between 1 and n, inclusive, which are relatively prime to n. We say that a positive integer k is total if $k = \frac{n}{\phi(n)}$, for some positive integer n. Find all total numbers.

13. Suppose that positive integers $a_1, a_2, ..., a_{2014}$ (not necessarily distinct) satisfy the condition that: $\frac{a_1}{a_2}, \frac{a_2}{a_3}, ..., \frac{a_{2013}}{a_{2014}}$ are pairwise distinct. What is the minimal possible number of distinct numbers in $\{a_1, a_2, ..., a_{2014}\}$?

14. A unitary divisor d of a number n is a divisor n that has the property $\gcd(d, \frac{n}{d}) = 1$. If $n = 1620$, what is the sum of all unitary divisors of d?

15. Let a, b, c be positive integers such that $\gcd(a, b) = 2$, $\gcd(b, c) = 3$, $\text{lcm}(a, c) = 42$, and $\text{lcm}(a, b) = 30$. Find abc.

2 Sources

1. 2013 Berkeley Math Tournament Spring Individual Problem 8
2. 2013 Berkeley Math Tournament Spring Individual Problem 16
3. 2013 Berkeley Math Tournament Spring Discrete Problem 1
4. 2013 Berkeley Math Tournament Spring Discrete Problem 4
5. 2013 Berkeley Math Tournament Spring Discrete Problem 5
6. 2013 Berkeley Math Tournament Spring Discrete Problem 7
7. 2013 Berkeley Math Tournament Spring Discrete Problem 10
8. 2014 Berkeley Math Tournament Fall Team Problem 18
9. 2014 Berkeley Math Tournament Fall Team Problem 10
10. 2014 Berkeley Math Tournament Fall Team Problem 9
11. 2014 Berkeley Math Tournament Spring Individual Problem 6
12. 2014 Berkeley Math Tournament Spring Discrete Problem 7
13. 2014 Berkeley Math Tournament Spring Discrete Problem 8
14. 2014 Berkeley Math Tournament Spring Team Problem 10
15. 2015 Berkeley Math Tournament Fall Individual Problem 12