Number Theory Handout #8 Answers and Solutions

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1 Answers

1. 5

2. 709

3. 12

4. 120

5. 10613

6. 52

7. 859

8. 1010527

9. 343

10. 0

11. $\frac{18}{49}$

12. -2013

13. 4

14. 5

15. 991

2 Solutions

1. When the integer $(\sqrt{3}+5)^{103}-(\sqrt{3}-5)^{103}$ is divided by 9, what is the remainder?

Solution: Notice that the given expression can be written as $2\cdot\sum_{n=0}^{51} 3^n 5^{103-2n}$. The remainder when this sum is divided by 9 is the same as the remainder when $2\cdot 3\cdot 5^{101}+2\cdot 5^{103}$ is divided by 9. By Euler's Totient Theorem, we know that $5^{103}\equiv 5^1\cdot 5^{6\cdot 17}\equiv 5^1\cdot 1\equiv 5 \pmod{9}$. Through brute force, we can find that $5^{101}\equiv 5^5\equiv 2 \pmod{9}$. It follows that the given sum is equivalent to $2\cdot 3\cdot 2+2\cdot 1\equiv 5 \pmod{9}$. It follows that our answer is $\boxed{5}$.

2. The value of 21! is $51,090,942,171,abc,440,000$ where a,b, and c are digits. What is the value of $100a+10b+c$?

Solution: Notice that 21! is divisible by each of $7,11$, and 13, and thus it is divisible by 1001. Therefore, the expression $0-440+\overline{abc}-171+942-90+51$ must be divisible by 1001. It follows that $292+\overline{abc}$ is divisible by 1001, and thus $\overline{abc}=\boxed{709}$.

3. Find the smallest two-digit positive integer that is a divisor of 201020112012.

Solution: Clearly the given number is not divisible by 10. Using the divisibility rule for 11, we can find that 201020112012 leaves a remainder of $2-1+0-2+1-1+0-2+0-1+0-2 \equiv 5 \pmod{11}$. Thus, the number is not divisible by 11. We can quickly find that the number is divisible by 3 by adding its digits, and we can find that the number is divisible by 4 by observing that its last two digits, 12, are divisible by 4. Thus, our answer is $3 \cdot 4 = \boxed{12}$.

4. When Meena turned 16 years old, her parents gave her a cake with n candles, where n has exactly 16 different positive integer divisors. What is the smallest possible value of n?

Solution: Notice that a number will only have exactly 16 positive integer factors if it is of one of the forms $pqrs, p^3qr, p^3q^3, p^7q$, or p^{15}. The smallest number of the form $pqrs$ is $2 \cdot 3 \cdot 5 \cdot 7 = 210$. The smallest number of the form p^3qr is $2^3 \cdot 3 \cdot 5 = 120$. The smallest number of the form p^3q^3 is $2^3 \cdot 3^3 = 216$. Thus, our answer is $\boxed{120}$.

5. The number $104,060,465$ is divisible by a five-digit prime number. What is that prime number?

Solution: We can notice that the given number is equivalent to $101^4+4\cdot2^4$. By the Sophie-Germain Factorization, it follows that the number can be factored as $(8+101^2-404)(8+101^2+404) = 9805 \cdot 10613$. It follows that our answer is $\boxed{10613}$.

6. Let N be the number of ordered pairs of integers (x, y) such that

$$4x^2 + 9y^2 \leq 1000000000.$$

Let a be the first digit of N (from the left) and let b be the second digit of N. What is the value of $10a + b$?

Solution: We can notice that for large areas, the number of lattice points inside the area is roughly the same as the area itself. The area of this ellipse is $\frac{\sqrt{1000000000}}{3} \cdot \frac{\sqrt{1000000000}}{2} \cdot \pi = \frac{1000000000\pi}{6}$. We can find that this is roughly 523000000. Therefore, we can reasonably say that $10a + b$ is $\boxed{52}$.

7. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, what is the value of $P(10)$?

Solution: Notice that if the constant term of the polynomial were divisible by any positive integer other than 1, then it would be impossible for the condition to be satisfied. Thus, the constant term is either 1 or -1. For our first case, assume the constant term is 1. If we let the quadratic be $P(n) = an^2 + bn + 1$, then $P(P(n))$ has a constant term of $a + b + 1$. Thus, for the same reason as above, we must have either $a + b = 0$ or $a + b = -2$. We know that $9a + 3b + 1 = 3(3a + b) + 1 = 89$. This is clearly impossible, as $89 - 1 = 88$ is not divisible by 3. Now assume the constant term is -1. Then the constant term of $P(P(n))$ is $a - b - 1$. It follows that either $a - b = 0$ or $a - b = 2$. We know that $9a + 3b - 1 = 89$, or $3a + b = 30$. If $a - b$ were 0, it would follow that $4a = 30$ which is impossible. Thus, $a - b = 2$, and $4a = 32$. It follows that $a = 8$ and $b = 6$, and our polynomial is $P(n) = 8n^2 + 6n - 1$. Thus, $P(10) = 800 + 60 - 1 = \boxed{859}$.

8. What is the least positive integer n such that $n!$ is a multiple of 2012^{2012}?

Solution: Notice that $2012^{2012} = 2^{4024} \cdot 5^{2012}$. Thus, we wish to find the smallest value of n such that $n!$ is a multiple of 503^{2012}. We can notice that $n = 503 \cdot 2012$ satisfies the property that $n!$ is divisible by $503^{2012+4} = 503^{2016}$. It follows that our answer is $n = 503 \cdot (2012 - 3) = 503 \cdot 2009 = \boxed{1010527}$

9. For how many ordered pairs of positive integers (x, y) is the least common multiple of x and y equal to $1,003,003,001$?

Solution: Notice that $1003003001 = 7^3 \cdot 11^3 \cdot 13^3$. For each prime power in this product, we have $2 \cdot 3 + 1 = 7$ possible ways to divide the factors of the corresponding prime to each number in the pair (x, y). Thus, our answer is $7^3 = \boxed{343}$.

10. When the binomial coefficient $\binom{125}{64}$ is written out in base 10, how many zeros are at the rightmost end?

Solution: By Legendre's Formula, 125! is divisible by 5^{31} and 2^{119}. Similarly, 64! is divisible by 5^{14} and 2^{63}. Similarly, 61! is divisible by 5^{14} and 2^{56}. It follows that $\binom{125}{64}$ is divisible by $5^{31-14-14} = 5^3$ and $2^{119-63-56} = 2^0$. Thus, there are $\boxed{0}$ zeroes at the end of $\binom{125}{64}$.

11. If n is a positive integer, let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n}.$$

Express your answer as a fraction in simplest form.

Solution: Notice that if $x = \frac{2}{9}$, then the given sum is equivalent to

$$\sum_{n=1}^{\infty} \frac{\phi(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} \phi(n)\left(\sum_{k=1}^{\infty} x^{kn}\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1,n|k}^{\infty} x^k\phi(n)\right) = \sum_{k=1}^{\infty} x^k\left(\sum_{n=1,n|k}^{\infty} \phi(n)\right)$$

It is well known that the sum $\sum_{n|k} \phi(n) = k$. It follows that our sum is now equivalent to

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} = \frac{x}{(1-x)^2}$$

It follows that the value of our sum is $\frac{\frac{2}{9}}{(\frac{7}{9})^2} = \boxed{\frac{18}{49}}$.

12. Say that an integer A is *yummy* if there exist several consecutive integers (including A) that add up to 2014. What is the smallest yummy integer?

Solution: Consider a set of n consecutive integers that add up to 2014. If the average of these integers is a, then we have that $na = 2014$. Because this is a set of consecutive integers, we must have that a is a positive integer multiple of $\frac{1}{2}$. It follows that n is maximized when $a = \frac{1}{2}$. In this case, we would have 4028 consecutive positive integers with an average of 0.5. This set would consist of all integers from -2013 to 2014. It follows that $\boxed{-2013}$ is the smallest yummy integer.

13. Say that an integer $n \geq 2$ is *delicious* if there exist n positive integers adding up to 2014 that have distinct remainders when divided by n. What is the smallest delicious integer?

Solution: If the remainders of these n integers are added up, then we would get $\frac{n(n-1)}{2}$. We can notice that this is divisible by n when n is odd, and when $n = 2x$, this is equivalent to $x \pmod{x}$. It follows that the smallest odd yummy integer is 19 and the smallest even yummy integer is $\boxed{4}$.

14. There are N students in a class. Each possible nonempty group of students selected a positive integer. All of these integers are distinct and add up to 2014. Compute the greatest possible value of N.

Solution: Notice that there are $2^N - 1$ distinct nonempty groups of students. The sum of these $2^N - 1 = a - 1$ distinct positive integers is at very least $\frac{a(a-1)}{2}$. It follows that the smallest possible value of a is $a = 32$, as any larger power of 2 will lead to a sum which is greater than 2014. It follows that the greatest possible value of N is $N = \boxed{5}$.

15. For how many integers k such that $0 \leq k \leq 2014$ is it true that the binomial coefficient $\binom{2014}{k}$ is a multiple of 4?

Solution: By Kummer's Theorem, the exponent of the largest power of 2 which divides $\binom{2014}{k}$ is equivalent to the number of carries when k is added to $2014-k$ in base 2. The binary representation of 2014 is 11111011110_2. If the number of carries in this sum is 0, the 2^0 and 2^5 digits in k must both be 0, but the other 9 digits can be arbitrary. It follows that there are $2^9 = 512$ values of k such that $\binom{2014}{k}$ is odd. In order for there to be 1 carry in this sum, Either k must have a 2^0 digit of 1 and a 2^1 digit of 0 as well as a 2^5 digit of 0 or k must have a 2^0 digit of 0 and a 2^5 digit of 1 as well as a 2^6 digit of 0. In this case, there are $2 \cdot 2^8 = 512$ additional values of k which make 2 the largest power of 2 which divides $\binom{2014}{k}$. It follows that there are $2014 + 1 - 1024 = \boxed{991}$ values of k such that $\binom{2014}{k}$ is divisible by 4.

3 Sources

1. Math Prize For Girls 2009 Problem 9
2. Math Prize For Girls 2009 Problem 18
3. Math Prize For Girls 2010 Problem 5
4. Math Prize For Girls 2010 Problem 8
5. Math Prize For Girls 2011 Problem 13
6. Math Prize For Girls 2011 Problem 16
7. Math Prize For Girls 2011 Problem 18
8. Math Prize For Girls 2012 Problem 3
9. Math Prize For Girls 2012 Problem 6
10. Math Prize For Girls 2013 Problem 2
11. Math Prize For Girls 2013 Problem 19
12. Math Prize For Girls 2014 Problem 4
13. Math Prize For Girls 2014 Problem 5
14. Math Prize For Girls 2014 Problem 6
15. Math Prize For Girls 2014 Problem 18