# Number Bases Handout Answers and Solutions

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## 1 Answers

**1.** 13

**2.** $131214_4$ or 13121

**3.** 4095

**4.** $14_6$ or 14

**5.** $661_{10}$ or 661

**6.** $2114_5$ or 2114

**7.** 13

**8.** $\frac{13}{4}$

**9.** $1135_8$ or 1135

**10.** 11

**11.** $\frac{16_b}{37_b}$

**12.** 7 or $7_{10}$

**13.** $240_{10}$ or 240

**14.** 3

**15.** 13

## 2 Solutions

**1.** In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers $b$ does the base-$b$-representation of 2013 end in the digit 3?

**Solution:** Notice that when a number is expressed in base $b$, its last digit is the same as the remainder when the number is divided by $b$. It follows that if $2013_{10} = ...3_b$, then $2013 = ab + 3$ for some integer $a$. It follows that $b$ is a factor of 2010 which is greater than 3. The prime factorization of 2010 is $2010 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 67^1$. It follows tht the number of factors of 2010 is $2^4 = 16$, and therefore the number of bases $b$ which work is $16 - 3 = \boxed{13}$.

**2.** Convert $731_8$ to a base-4 number.

**Solution:** We will begin by converting $731_8$ to binary, and then we will convert the result to base

4. Letting each original digit represent three digits of the binary result, we get the binary form $111011001_2$. Now, letting each base 4 digit represent 2 binary digits, we get the final base 4 result of $\boxed{131214_4}$.

**3.** What is the largest base 10 number that can be expressed in 3 digits of base 16?

**Solution:** Notice that the smallest 4-digit number in base 16 is $1000_{16} = 16^3$. It follows that the largest base 10 number which can be expressed with 3 base 16 digits is $16^3 - 1 = \boxed{4095}$.

**4.** What is the base 6 representation of the base 10 number 10?

**Solution:** By simple base conversion, we can rewrite $10_{10} = 6_{10} + 4_{10} = 10_6 + 4_6 = \boxed{14_6}$.

**5.** What is the base 8 number 1225 in base 10?

**Solution:** We can rewrite the number as $1225_8 = 1000_8 + 200_8 + 20_8 + 5_8 = 8^3 + 2 \cdot 8^2 + 2 \cdot 8^1 + 5 = \boxed{661_{10}}$.

**6.** Evaluate $11_2 + 111_3 + 1111_4$ and express as a base 5 number.

**Solution:** We will begin by converting each number to base 10. This can be simplified by rewriting the result as $11_2 + 111_3 + 1111_4 = 100_2 + 1000_3 + 10000_4 - 3 = 2^2 + 3^3 + 4^4 - 3 = 284_{10}$. Then we can rewrite this result as $284_{10} = 2 \cdot 5^3 + 5^2 + 5^1 + 4 = \boxed{2114_5}$.

**7.** In what base $b$ with $b > 7$, is 3 times 7 equal to 18?

**Solution:** We will convert all numbers to base 10. This gives us the equation $3 \cdot 7 = b + 8$. Solving, we get $b = \boxed{13}$.

**8.** When 2017 (in base 10) is expressed in base 7, what is the base-10 average of those base-7 digits?

**Solution:** We can rewrite 2017 as $2017_{10} = 5 \cdot 7^3 + 6 \cdot 7^2 + 7^1 + 1 = 5611_7$. The sum of these base 7 digits is $5 + 6 + 1 + 1 = 13_{10}$, and the base-10 average of these digits is $\boxed{\frac{13}{4}}$.

**9.** $N$ is the smallest positive integer for which $275 \cdot N$ (base 10) is a perfect cube. Write $N$ in octal.

**Solution:** The prime factorization of 275 is $275 = 5^2 \cdot 11^1$. It follows that the smallest value of $N$ such that $275N$ is a perfect cube is $N = 5^1 \cdot 11^2 = 605$. It follows that $N$ can be rewritten as $N = 605 = 8^3 + 8^2 + 3 \cdot 8^1 + 5 = \boxed{1135_8}$.

**10.** Let $A, B, C$ be digits, $x_{10} = ABC_4$, and $y_{10} = ABC_6$. If $y = 2x$, and the sum of all possible values of $x$ is $z$, find the sum of the digits of $z$.

**Solution:** Converting everything to base 10, we get $x = 16A + 4B + C$ and $y = 36A + 6B + C$. It follows that $32A + 8B + 2C = 36A + 6B + C$, or $2B + C = 4A$. Because $A, B, C < 4$, the only triples $(A, B, C)$ which work are $(A, B, C) = (1, 2, 0), (1, 1, 2)$, and $(2, 3, 2)$. These correspond to $x$-values of 24, 22, and 46, respectively. It follows that $z = 24 + 22 + 46 = 92$, and the sum of the digits of $z$ is $\boxed{11}$.

**11.** The fraction $\frac{103_b}{136_b}$ can be reduced to $\frac{14_b}{18_b}$ (which is simplest since it is irreducible in base 10). What is the simplest form of the fraction $\frac{149_b}{338_b}$, written as a quotient of integers, each in base $b$?

**Solution:** Notice that the given statement is equivalent to $33_b \cdot 18_b = 4b \cdot 136_b$. Converting everything to base $b$, we get $(3b + 3)(b + 8) = 4b^2 + 12b + 24$. Simplifying, we get $b^2 - 15b = 0$, or $b = 15$. It follows that $149_b = 149_{15} = 225 + 60 + 9 = 294$, and $338_b = 338_{15} = 675 + 45 + 8 = 728$.

It follows that the simplest form of the fraction is $\frac{21}{52}$ in base 10, and converting this to base $b = 15$, we get the fraction $\boxed{\dfrac{16_b}{37_b}}$.

**12.** When the decimal number 271 is converted to base 4, what is the sum of the digits of the base four number, expressed in base ten?

**Solution:** We can rewrite $271 = 4^4 + 3 \cdot 4 + 3 = 10033_4$. It follows that the sum of the digits of this base four number is $1 + 3 + 3 = \boxed{7}$.

**13.** In base eleven, the letter $A$ is the symbol for 10. What is $1A9_{11}$ expressed in base ten?

**Solution:** We can rewrite this as $1A9_{11} = 11^2 + 10 \cdot 11^1 + 9 = \boxed{240_{10}}$.

**14.** The number $A$, when expressed in base 7, is $0.222..._7$. When expressed in base 10, $A$ can be written as $0.ppp..._{10}$. What is the value of $p$?

**Solution:** We can remember that $0.\overline{x}_b$ is equivalent to $\frac{x}{b-1}$. It follows that the given expression is $\frac{2}{6} = \frac{1}{3}$, and therefore $p = \boxed{3}$.

**15.** The four digit number $X45Y_{12}$ is divisible by $143_{10}$. Evaluate $X + Y$ in base 10.

**Solution:** Remembering that a base $n$ number will only be divisible by $n - 1$ if the sum of its digits is divisible by $n - 1$, and that 143 is divisible by $12 - 1 = 11$, We know that $X + Y + 9$ has to be divisible by 11. It follows that either $X + Y = 13$ or $X + Y = 2$ as $X, Y < 12$. Similarly, we know that a base $n$ number will only be divisible by $n + 1$ if the alternating sum of its digits is divisible by $n + 1$. It follows that $X - Y + 1$ is divisible by 13. Notice that if $X + Y = 2$, we must either have $(X, Y) = (2, 0)$ or $(X, Y) = (1, 1)$. Because neither of these pairs satisfy $X - Y + 1$ is divisible by 13, we must have $(X, Y) = (6, 7)$ and $X + Y = \boxed{13}$.

## 3 Sources

**1.** 2013 AMC 10A Problem 23
**2.** 2014-2015 Log1 Contest Theta Individual Problem 3
**3.** 2007-2008 Log1 Contest Theta Individual Problem 7
**4.** 2008-2009 Log1 Contest Theta Number Theory Problem 5
**5.** 2008-2009 Log1 Contest Theta Number Theory Problem 9
**6.** 2011-2012 Log1 Contest Theta Number Theory Problem 4
**7.** 2016-2017 National Mu Alpha Theta Open Number Theory Problem 1
**8.** 2016-2017 National Mu Alpha Theta Open Number Theory Problem 6
**9.** 2016-2017 National Mu Alpha Theta Open Number Theory Problem 15
**10.** 2015-2016 National Mu Alpha Theta Open Number Theory Problem 9
**11.** 2015-2016 National Mu Alpha Theta Open Number Theory Problem 14
**12.** 2014-2015 National Mu Alpha Theta Open Number Theory Problem 5
**13.** 2014-2015 National Mu Alpha Theta Open Number Theory Problem 8
**14.** 2014-2015 National Mu Alpha Theta Open Number Theory Problem 18
**15.** 2012-2013 National Mu Alpha Theta Open Number Theory Problem 22