Number Theory Handout 2 Answers and Solutions

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1 Answers

1. 5

2. 9

3. 16

4. 167

5. 960

6. 2067

7. 978

8. 604

9. $\{83, 97\}$

10. 28

11. 1956

12. 630

13. 781

14. $\frac{1}{2}$

15. 310

2 Solutions

1. A boy is standing in the middle of a very very long staircase and he has two pogo sticks. One pogo stick allows him to jump 220 steps up the staircase. The second pogo stick allows him to jump 125 steps down the staircase. What is the smallest positive number of steps that he can reach from his original position by a series of jumps?

Solution: Notice that regardless of which move the boy makes, his distance from his original location will always be a multiple of $\gcd(125, 220) = \boxed{5}$ steps. He can get 5 steps away from his original position by jumping up 4 times and jumping down 7 times as $4 \cdot 220 - 7 \cdot 125 = 5$ as desired.

2. Canada gained partial independence from the United Kingdom in 1867, beginning its long role as the headgear of the United States. It gained its full independence in 1982. What is the last digit of 1867^{1982}?

Solution 1: Notice that we only care about the units digit so this question is equivalent to determining the units digit of $1867^{1982} \equiv 7^{1982} \pmod{10}$. Observing the units digits of the first few powers of 7, we notice that their units digits repeat in the pattern $7, 9, 3, 1, 7, 9, 3, 1, \cdots$. It follows that our answer is $7^{1982} \equiv (7^4)^{495} \cdot 7^2 \equiv 1^{495} \cdot 7^2 \equiv \boxed{9} \pmod{10}$ as desired.

Note: This solution uses properties of modular arithmetic. If you have not seen modular arithmetic before, check this link: **Modular Arithmetic: AoPS**.

Solution 2: By Euler's Totient Theorem, we have $1867^{1982} \equiv 1867^{1982 \pmod{\phi(10)}} \equiv 1867^2 \equiv 7^2 \equiv \boxed{9} \pmod{10}$ as desired.

3. Kim, who has a tragic allergy to cake, is having a birthday party. She invites 12 people but isnt sure if 11 or 12 will show up. However, she needs to cut the cake before the party starts. What is the least number of slices (not necessarily of equal size) that she will need to cut to ensure that the cake can be equally split among *either* 11 or 12 guests with no excess?

Solution: The optimal method for these divisions to be possible is to cut the cake into 12 equal pieces with 6 slices through the center of the cake and then cutting one of those 12 pieces into 11 equal parts with 10 more slices through the middle. If 12 people show up, 11 people each get one of the big pieces and the last person gets all of the small pieces. If 11 people show up, each person gets one big piece and one small piece. Therefore, our answer is $10 + 6 = \boxed{16}$.

Note: According to the Berkeley Math Tournament website, the answer to this question is 22. However, as shown above, the question can be done with only 16 cuts.

4. Given that 2012022012 has 8 distinct prime factors, find its largest prime factor.

Solution: Notice that we can rewrite this number as $2012 \cdot 1000001 + 20000 = 2002 \cdot (1000001) + 10 \cdot (1000^2 + 2000 + 1) = 1001 \cdot (2000002 + 10010) = 1001 \cdot (2010012)$. Notice that 2010012 ends in 12, so it is divisible by 4. In addition, its sum of digits is 6 so it is divisible by 3. Therefore, the original number is $2012022012 = 1001 \cdot 4 \cdot 3 \cdot 167501$. From here, we can notice that $501 = 3 \cdot 167$, so we know that $167501 = 1003 \cdot 167$. Finally, we can notice that $1020 - 17 = 17 \cdot 60 - 17 = 17 \cdot 59 = 1003$. Therefore, the prime factorization of 2012022012 is $2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 59 \cdot 167$. Therefore, our answer is $\boxed{167}$ as desired.

5. Find the smallest number with exactly 28 divisors.

Solution: Remembering that the number of divisors is the product of all the values obtained when adding 1 to the exponents in the prime factorization of the number. It follows that a number with 28 divisors must be of the form p^{27}, $p^{13}q$, p^6q^3, or p^6qr, where p, q, and r are arbitrary distinct prime numbers. Clearly, the last two forms will produce the smallest values. The smallest number of the form p^6q^3 is $2^63^3 = 1728$ and the smallest number of the form p^6qr is $2^63^15^1 = 960$. Therefore, our answer is $\boxed{960}$.

6. Let a, b, c, d, $(a + b + c + 18 + d)$, $(a + b + c + 18 - d)$, $(b + c)$, and $(c + d)$ be distinct prime numbers such that $a + b + c = 2010$, $a, b, c, d \neq 3$, and $d \leq 50$. Find the maximum value of the difference between two of these prime numbers.

Solution: Notice that because $a + b + c = 2010$ is even, one of a, b, c must be even, and because they are prime, one of them must be 2. If a were 2, $b + c$ would equal 2008, which is not prime. If b were 2, then c would be an odd prime, and the only way $c + d$ could be prime is if d were 2, in which case $b + c$ and $c + d$ would not be distinct. Therefore we know $c = 2$. We know that $2028 + d$ and $2028 - d$ are both prime and $d \neq a, b, c$. Clearly to maximize the difference between two of these prime numbers, we want $2028 + d - 2 = 2026 + d$ to be maximized, given $d \leq 50$. By brute force, we can find that the maximum value of d which satisfies $d, d + 2, 2028 + d$, and $2028 - d$ are all prime is $d = 41$ as $41, 43, 2069$, and 1987 are all prime. We could speed up this brute force by noticing d must be $5 \pmod 6$ for d and $d+2$ to both be prime. We can check that this works by noticing that the quadruple $(a, b, c, d) = (1997, 11, 2, 41)$ satisfies the given condition. Therefore, our answer is $2026 + 41 = \boxed{2067}$ as desired.

7. Let S be the set of all rational numbers $x \in [0, 1]$ with repeating base 6 expansion $x = 0.\overline{a_1 a_2 \cdots a_k} = 0.a_1 a_2 \cdots a_k a_1 a_2 \cdots a_k \cdots$ for some finite sequence $\{a_i\}_{i=1}^{k}$ of distinct nonnegative integers less than 6. What is the sum of all numbers that can be written in this form? (Put your answer in base 10.)

Solution: We can notice that the number $0.\overline{a_1 a_2 \cdots a_k}_6 = \frac{\overline{a_1 a_2 \cdots a_k}_6}{6^k - 1}$. From here we can do casework on the value of k.

Case 1: $k = 1$.

In this case, the numerator can take any value from 0 to 5. Therefore our answer is $\frac{5 \cdot 6}{2 \cdot (6^1 - 1)} = 3$.

Case 2: $k = 2$.

In this case, for each digit from 0 to 5, there are $6 - 1 = 5$ distinct possibilities where that digit is the value of a_1 and 5 possibilities where that digit is the value of a_2. Because we are working in base 6, the value of a_1 is multiplied by 6 and our expression, so the sum of all numbers in this case is $\frac{5 \cdot 6 \cdot (5 \cdot 6 + 5)}{2 \cdot (6^2 - 1)} = 15$.

Case 3: $k = 3$.

Using similar logic to the above case, we find that the sum of the numbers in this case is $\frac{5 \cdot 6 \cdot (5 \cdot 4 \cdot 36 + 5 \cdot 4 \cdot 6 + 5 \cdot 4)}{2 \cdot (6^3 - 1)} = 60$.

Case 4: $k = 4$.

Using similar logic to the above case, we find that the sum of the numbers in this case is $\frac{5 \cdot 6 \cdot (5 \cdot 4 \cdot 3 \cdot 216 + 5 \cdot 4 \cdot 3 \cdot 36 + 5 \cdot 4 \cot 3 \cdot 6 + 5 \cdot 4 \cdot 3)}{2 \cdot (6^4 - 1)} = 180$.

Case 5: $k = 5$.

Using similar logic to the above case, we find that the sum of the numbers in this case is $\frac{5 \cdot 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1296 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 216 + 5 \cdot 4 \cot 3 \cdot 2 \cdot 36 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 6 + 5 \cdot 4 \cdot 3 \cdot 2)}{2 \cdot (6^5 - 1)} = 360$.

Case 6: $k = 6$.

Using similar logic to the above case, we find that the sum of the numbers in this case is $\frac{5 \cdot 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 7776 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1296 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 216 + 5 \cdot 4 \cot 3 \cdot 2 \cdot 36 + 5 \cdot 4 \cdot 3 \cdot 2 \cdot 6 + 5 \cdot 4 \cdot 3 \cdot 2)}{2 \cdot (6^6 - 1)} = 360$.

Adding up all of our cases, we get a total sum of $3 + 15 + 60 + 180 + 360 + 360 = \boxed{978}$ as desired.

8. Let $p > 1$ be relatively prime to 10. Let n be any positive number and d be the last digit of n. Define $f(n) = \lfloor \frac{n}{10} \rfloor + d \cdot m$. Then, we can call m a divisibility multiplier for p, if $f(n)$ is divisible by p if and only if n is divisible by p. Find a divisibility multiplier for 2013.

Solution: Let $n = 10x + m$. We wish to find a value m such that $x + md \equiv 0 \pmod{2013}$ if and only if $10x + m \equiv 0 \pmod{2013}$. If $10x + m \equiv 0 \pmod{2013}$, then we know $m \equiv -10x \pmod{2013}$. It follows that $x + md \equiv x(1 - 10d) \equiv 0 \pmod{2013}$. In order this to follow for all values of x and n, we must have $1 - 10d \equiv 0 \pmod{2013}$. It follows that $d \equiv 10^{-1} \pmod{2013}$. By inspection we can notice that $10 \cdot 604 = 6040 \equiv 1 \pmod{2013}$, and it follows that $d = \boxed{604}$ is a divisibility multiplier for 2013.

9. Find all prime factors of 8051.

Solution: We can notice by inspection that $8051 - 8100 - 49 = 90^2 - 7^2 = (90+7)\cdot(90-7) = 97\cdot 83$. Therefore, our answer is $\boxed{\{83, 97\}}$ as desired.

10. What is the smallest positive n so that $17^n + n$ is divisible by 29?

Solution: Listing remainders when powers of 17 are divided by 29, we get the sequence $17, 28, 12, 1, 17, 28, 12, 1, \cdots$. It follows that values of n that work are numbers which are congruent to $1 \pmod 4$ and $12 \pmod{29}$, congruent to $2 \pmod 4$ and $1 \pmod{29}$, congruent to $3 \pmod 4$ and $17 \pmod{29}$, or congruent to $0 \pmod 4$ and $28 \pmod{29}$. The smallest number which satisfies one of these four possibilities is $\boxed{28}$ as desired.

11. What is the largest integer n so that $\dfrac{n^2 - 2012}{n+7}$ is also an integer?

Solution: We know that $\frac{n^2-49}{n+7} = n - 7$ is an integer, so $\frac{n^2-2012}{n+7}$ will only be an integer if $\frac{n^2-49}{n+7} - \frac{n^2-2012}{n+7} = \frac{1963}{n+7}$ is an integer. The largest value of n which satisfies $n+7$ divides 1963 is $1963 - 7 = \boxed{1956}$.

12. Let $\phi(n)$ be the Euler totient function. What is the sum of all n for which $\frac{n}{\phi(n)}$ is maximal for $1 \le n \le 500$?

Solution: Because $f(n) = n$ and $\phi(n)$ are both multiplicative functions, we know $g(n) = \frac{n}{\phi(n)}$ is also a multiplicative function. We can determine that $g(p^n) = \frac{p}{p-1}$. It follows that to maximize $g(n)$, n should have as many small prime factors as possible. The largest number of distinct prime factors that a number less than 500 can have is 4, with the smallest value being $2 \cdot 3 \cdot 5 \cdot 7 = 210$. This number also has the 4 smallest prime factors possible, so it will have the maximum value of $g(n)$ among all numbers less than or equal to 500. However, $2 \cdot 210 = 420$ will also have the maximum value of $g(n)$, so our answer is $210 + 420 = \boxed{630}$ as desired.

13. Let n be the number so that $1 - 2 + 3 - 4 + \cdots - (n-1) + n = 2012$. What is $4^{2012} \pmod n$?

Solution: If we let $n = 2x + 1$ be odd, then the value of $1 - 2 + 3 - 4 + \cdots - (n-1) + n$ is $x \cdot (-1) + 2x + 1 = x + 1$. If this is equal to 2012, then $x = 2011$ and $n = 4023$. Notice that $4^{2012} \equiv (2^2)^{2012} = 2^{4024} \pmod{4023}$. Factorizing 4023, we get $4023 = 3^3 \cdot 149$. Therefore, if we could find $2^{4024} \pmod{27}$ and $2^{4024} \pmod{149}$, we could combine the results to get $2^{4024} \pmod{4023}$ by the Chinese Remainder Theorem. Using Euler's Totient Theorem, we can find $2^{4024} \equiv 2^{4024 \pmod{\phi(27)}} \equiv 2^{10} \pmod{27}$. We can manually compute this to be 25. Similarly, we can use Euler's Totient Theorem to find $2^{4024} \equiv 2^{4024 \pmod{\phi(149)}} \equiv 2^{28} \pmod{149}$. We can rewrite this as $(2^7)^4 \equiv (-21)^4 \equiv (-6)^2 \equiv 36 \pmod{149}$. Therefore, we wish to find a number y less than 4023 that satisfies $y \equiv 36 \pmod{149}$ and $y \equiv 25 \pmod{27}$. Brute forcing, we can get $y = \boxed{781}$ satsifies these conditions.

Note: In the official solution to this problem on the Berkeley Math Tournament Website, the solution falsely claims $128^4 \equiv (-20)^4 \pmod{149}$, leading to a different answer. I have corrected the mistake in this solution, and checking with Wolfram Alpha confirms that the answer is 781.

14. Let $\phi(n)$ be the Euler totient function, and let $S = \left\{ x \mid \frac{x}{\phi(x)} = 3 \right\}$. What is $\sum_{x \in S} \frac{1}{x}$?

Solution: Notice that $g(x) = \frac{x}{\phi(x)}$ is multiplicative and that $g(p^n) = \frac{p}{p-1}$. Computing this value for small prime powers gives us $g(2) = 2$, $g(3) = \frac{3}{2}$, $g(5) = \frac{5}{4}$, $g(7) = \frac{7}{6}$, and so on. We can notice that because $\frac{x}{\phi(x)}$ is 3, 3 is a factor of x. Therefore, if we let the largest factor of x which is not a multiple of 3 be y, we have $g(y) = 2$. We can notice that if y is not a power of 2, then $g(y)$ must have some prime factor in its numerator which is greater than 2. Therefore, x must be a power of 3 multiplied by a power of 2. Therefore, our answer is $(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots)\cdot(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots) = 1\cdot\frac{1}{2} = \boxed{\frac{1}{2}}$

15. Denote $f(N)$ as the largest odd divisor of N. Compute $f(1)+f(2)+f(3)+\cdots+f(29)+f(30)$.

Solution: Let $g(n) = f(1)+f(2)+\cdots+f(n)$. We wish to compute $g(30)$. We can notice that this is equivalent to $g(30) = 1+3+5+\cdots+29+g(15)$, as we can divide every even number in this range by 2 to get a number with an equivalent value of f. Continuing with this recursive nature, we get $g(30) = 1+3+\cdots+29+1+3+\cdots+15+1+3+\cdots+7+1+3+1$. Remembering that the sum of the first n odd integers is n^2, we can find that this is equal to $15^2+8^2+4^2+2^2+1^2 = \boxed{310}$.

3 Sources

1. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 2
2. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 6
3. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 16
4. 2012 Berkeley Math Tournament Fall 2012 Team Problem 10
5. 2012 Berkeley Math Tournament Spring 2012 Individual Problem 2
6. 2012 Berkeley Math Tournament Spring 2012 Individual Problem 7
7. 2012 Berkeley Math Tournament Spring 2012 Team Problem 1
8. 2012 Berkeley Math Tournament Spring 2012 Team Problem 5
9. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 1 Problem 1
10. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 1 Problem 4
11. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 2 Problem 4
12. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 4 Problem 5
13. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 5 Problem 1
14. 2012 Berkeley Math Tournament Spring 2012 Tournament Consolation Round Problem 3
15. 2012 Berkeley Math Tournament Spring 2012 Tournament Consolation Round Problem 4