Number Theory Handout 2

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1 Problems

1. A boy is standing in the middle of a very very long staircase and he has two pogo sticks. One pogo stick allows him to jump 220 steps up the staircase. The second pogo stick allows him to jump 125 steps down the staircase. What is the smallest positive number of steps that he can reach from his original position by a series of jumps?

2. Canada gained partial independence from the United Kingdom in 1867, beginning its long role as the headgear of the United States. It gained its full independence in 1982. What is the last digit of 1867^{1982}?

3. Kim, who has a tragic allergy to cake, is having a birthday party. She invites 12 people but isnt sure if 11 or 12 will show up. However, she needs to cut the cake before the party starts. What is the least number of slices (not necessarily of equal size) that she will need to cut to ensure that the cake can be equally split among *either* 11 or 12 guests with no excess?

4. Given that 2012022012 has 8 distinct prime factors, find its largest prime factor.

5. Find the smallest number with exactly 28 divisors.

6. Let a, b, c, d, $(a + b + c + 18 + d)$, $(a + b + c + 18 - d)$, $(b + c)$, and $(c + d)$ be distinct prime numbers such that $a + b + c = 2010$, $a, b, c, d \neq 3$, and $d \leq 50$. Find the maximum value of the difference between two of these prime numbers.

7. Let S be the set of all rational numbers $x \in [0, 1]$ with repeating base 6 expansion $x = 0.\overline{a_1a_2\cdots a_k} = 0.a_1a_2\cdots a_ka_1a_2\cdots a_k\cdots$ for some finite sequence $\{a_i\}_{i=1}^{k}$ of distinct nonnegative integers less than 6. What is the sum of all numbers that can be written in this form? (Put your answer in base 10.)

8. Let $p > 1$ be relatively prime to 10. Let n be any positive number and d be the last digit of n. Define $f(n) = \lfloor \frac{n}{10} \rfloor + d \cdot m$. Then, we can call m a divisibility multiplier for p, if $f(n)$ is divisible by p if and only if n is divisible by p. Find a divisibility multiplier for 2013.

9. Find all prime factors of 8051.

10. What is the smallest positive n so that $17^n + n$ is divisible by 29?

11. What is the largest integer n so that $\dfrac{n^2 - 2012}{n + 7}$ is also an integer?

12. Let $\phi(n)$ be the Euler totient function. What is the sum of all n for which $\frac{n}{\phi(n)}$ is maximal for $1 \leq n \leq 500$?

13. Let n be the number so that $1 - 2 + 3 - 4 + \cdots - (n - 1) + n = 2012$. What is $4^{2012} \pmod{n}$?

14. Let $\phi(n)$ be the Euler totient function, and let $S = \left\{x \mid \frac{x}{\phi(x)} = 3\right\}$. What is $\sum_{x \in S} \frac{1}{x}$?

15. Denote $f(N)$ as the largest odd divisor of N. Compute $f(1)+f(2)+f(3)+\cdots+f(29)+f(30)$.

2 Sources

1. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 2
2. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 6
3. 2012 Berkeley Math Tournament Fall 2012 Individual Problem 16
4. 2012 Berkeley Math Tournament Fall 2012 Team Problem 10
5. 2012 Berkeley Math Tournament Spring 2012 Individual Problem 2
6. 2012 Berkeley Math Tournament Spring 2012 Individual Problem 7
7. 2012 Berkeley Math Tournament Spring 2012 Team Problem 1
8. 2012 Berkeley Math Tournament Spring 2012 Team Problem 5
9. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 1 Problem 1
10. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 1 Problem 4
11. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 2 Problem 4
12. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 4 Problem 5
13. 2012 Berkeley Math Tournament Spring 2012 Tournament Round 5 Problem 1
14. 2012 Berkeley Math Tournament Spring 2012 Tournament Consolation Round Problem 3
15. 2012 Berkeley Math Tournament Spring 2012 Tournament Consolation Round Problem 4