Number Theory Handout 3 Answers and Solutions

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1 Answers

1. $-1, 0, 1$

2. 35

3. 1

4. 28

5. 2012^{2011}

6. 24

7. 66

8. 105

9. 15

10. 50

11. 32400

12. 4

13. 60

14. 0

15. 1536

2 Solutions

1. We say s grows to r if there exists some integer $n > 0$ such that $s^n = r$. Call a real number r sparse if there are only finitely many real numbers s that grow to r. Find all real numbers that are sparse.

Solution: Notice that if $r > 1$ or $r < -1$, then for any odd integer n greater than 0, there will exist a unique real number s such that $s^n = r$. Similarly, when $0 < r < 1$ or $-1 < r < 0$, any odd integer n will produce a unique real value of s with this property. Notice that when $r = 0$, s must equal 0 for s to grow to r. When $r = 1$, s must equal 1 or -1 for s to grow to r. Finally, when $r = -1$, s must equal -1 for s to grow to r. Therefore, the only sparse numbers are $\boxed{-1, 0, 1}$

2. How many integers between 2 and 100 inclusive cannot be written as $m \cdot n$, where m and n have no common factors and neither m nor n is equal to 1? Note that there are 25 primes less than 100.

Solution: Notice that in order for this condition to hold, the integer x with this property must be the product of at least two distinct prime powers. The only numbers which are not of this form are 1, the primes, and the prime powers. We are given that there are 25 primes, and the only prime powers less than 100 are $4, 8, 16, 32, 64, 9, 27, 81, 25,$ and 49. Therefore, our answer is $25 + 10 = \boxed{35}$ as desired.

3. Determine the remainder when

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}$$

is divided by 7.

Solution: Notice that $2^3 = 8 \equiv 1 \pmod 7$. It follows that the remainder when 2^x is divided by 7 only depends on the remainder when x is divided by 3. Notice that $\frac{n(n+1)}{2}$ is divisible by 3 when $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$, and it leaves a remainder of 1 when divided by 3 otherwise. It follows that the sum of any 3 consecutive powers of the form $2^{\frac{n(n+1)}{2}}$ leaves a remainder of $2^1 + 2^0 + 2^0 = 4$. Notice that in this sum, we have 2011 consecutive powers. We can rewrite this as the first term added to $\frac{2010}{3} = 670$ groups of 3 consecutive powers. It follows that the remainder when the sum is divided by 7 is the same as the remainder when $670 \cdot 4 + 2$ is divided by 7. We can calculate that this leaves a remainder of $\boxed{1}$ when divided by 7.

4. What is the sum of all of the distinct prime factors of $25^3 - 27^2$?

Solution: Notice that we can rewrite this as $5^6 - 3^6$. This can be rewritten as $(125+27)(125-27) = 152 \cdot 98 = 2^4 \cdot 7^2 \cdot 19^1$. Therefore, the sum of its prime factors is $2 + 7 + 19 = \boxed{28}$.

5. Find the number of ordered 2012-tuples of integers $(x_1, x_2, ..., x_{2012})$, with each integer between 0 and 2011 inclusive, such that the sum $x_1 + 2x_2 + 3x_3 + \cdots + 2012x_{2012}$ is divisible by 2012.

Solution: Notice that no matter what the values of $x_2, x_3, \cdots x_{2012}$ are, there is exactly one value of x_1 which makes the sum divisible by 2012. Therefore, the number of 2012-tuples with this property is the number of choices for the values of x_2 through x_{2012}, which is $\boxed{2012^{2011}}$

6. What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

Solution: The smallest two numbers which are the products of four not necessarily distinct prime numbers are $2^4 = 16$ and $2^3 \cdot 3^1 = 24$. Because 16 is a perfect square, our answer is $\boxed{24}$.

7. Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

Solution: Notice that a number will leave a remainder of 1 when divided by 3 if and only if the number of factors in its prime factorization which leave a remainder of 2 when divided by 3 is even and the number is not divisible by 3. By brute force and Legendre's Formula, we have that $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$. It follows that a factor which leaves a remainder of 1 when divided by 3 can have any arbitrary number of factors of 7 between 0 and 1 and an even number of other prime factors. If the number has an even number of factors of 2, then the number of possibilities is $6 \cdot 3 \cdot 2 \cdot 1 = 36$. If the number has an odd number of factors of 2, then the number of possibilities is $5 \cdot 3 \cdot 2 = 30$. It follows that the total number of divisors with this property is $36 + 30 = \boxed{66}$.

8. How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, ...$?

Solution: Notice that any number which can be expressed as the sum of distinct powers of 3 is the base 3 interpretation of a binary string. 1000 in base 3 is 1101001, and when interpreted as a binary number, this number has a value of $2^6 + 2^5 + 2^3 + 2^0 = \boxed{105}$. Because every binary number below 1101001 also corresponds to a number with this property, this

9. Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

Solution: Notice that we can rewrite these numbers as $2^{16} - 1$ and $2^{36} - 1$, respectively. Using the Euclidean Algorithm, we can determine that $\gcd(2^{36} - 1, 2^{16} - 1) = \gcd(2^{36} - 1 - (2^{36} - 2^{20}), 2^{16} - 1) = \gcd(2^{20} - 1, 2^{16} - 1)$. Similarly, $\gcd(2^{20} - 1, 2^{16} - 1) = \gcd(2^{20} - 1 - (2^{20} - 2^4), 2^{16} - 1) = \gcd(2^4 - 1, 2^{16} - 1)$. From here, we can use Difference of Fourth Powers to find that $2^{16} - 1$ is divisible by $2^4 - 1$. It follows that the greatest common divisor of these two numbers is $2^4 - 1 = \boxed{15}$ as desired.

Remark: Using the Euclidean Algorithm in a similar manner, we can prove the more general fact that $\gcd(2^x - 1, 2^y - 1) = 2^{\gcd(x,y)} - 1$.

10. Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \leq n \leq 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

Solution: We can observe that the remainder when a_x is divided by 100 will become constant even for small values of x regardless of the value of n. (In other words, $a_x = a_{x+1} = a_{x+2} = \cdots$ for some small value of x). Notice that for large values of x, a_x is always divisible by 4, and its remainder when divided by 25 depends on the remainder when a_{x-1} is divided by 20. Clearly, a_{x-1} is divisible by 4, and it is clearly a power of 16, so $a_{x-1} \equiv 1 \pmod 5$ and $a_{x-1} \equiv 0 \pmod 4$ meaning $a_{x-1} \equiv 16 \pmod{20}$. It follows that $a_x \equiv 2^{16} \pmod{25}$. We can compute that this is equivalent to $2^{16} \equiv 11 \pmod{25}$. Therefore, $a_{1000} \equiv 0 \pmod 4$ and $a_{1000} \equiv 4 \pmod{25}$. It follows that $a_{1000} \equiv 36 \pmod{100}$ by the Chinese Remainder Theorem. Now, we must find all integers n such that $2^n \equiv 36 \pmod{100}$. Notice that $2^n \equiv 0 \pmod 4$. Therefore, we only need to find the values of n such that $2^n \equiv 11 \pmod{25}$. We know that the order of $2 \pmod{25}$ divides $\phi(25) = 20$ We can compute that 2^{10} and 2^4 do not leave remainders of $1 \pmod{25}$, so it follows that n must satisfy some residue $\pmod{20}$. However, we know that $2^{16} \equiv 11 \pmod{25}$, so it follows that $n \equiv 16 \pmod{20}$. It follows that the number of integers with this property is $\frac{1000}{20} = \boxed{50}$ as desired.

11. Five guys each have a positive integer (the integers are not necessarily distinct). The greatest common divisor of any two guys numbers is always more than 1, but the greatest common divisor of all the numbers is 1. What is the minimum possible value of the product of the numbers?

Solution: Let the five numbers be a, b, c, d, and e. To minimize the product of these numbers, we want to make the greatest common divisors of pairs of numbers as small as possibly. Therefore, it makes sense to make a, b, c, d divisible by 2 and e not divisible by 2. It also makes sense to make b, c, d, e divisible by 3 and e not divisible by 3. Finally, to make $\gcd(a, e)$ greater than 1, it makes sense to make a and e divisible by 5. It follows that the minimum value of $abcde$ is $2^4 \cdot 3^4 \cdot 5^2 = 180^2 = \boxed{32400}$.

12. Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^{4^{3^{2^1}}}}}$ repeatedly until a single base 12 digit remains. Find this digit.

Solution: Notice that in base 10, if we replace a number with the sum of its digits, its remainder when divided by $10 - 1 = 9$ does not change. A similar feature exists in base 12 where if we replace a number with the sum of its digits, its remainder when divided by $12 - 1 = 11$ does not change. Therefore, we need to find the remainder when $7^{6^{5^{\cdot^{\cdot}}}}$ is divided by 11. This depends on the remainder when $6^{5^{\cdot^{\cdot}}}$ is divided by $\phi(11) = 10$. However, $6^{5^{\cdot^{\cdot}}}$ must leave a remainder of 6 when divided by 10 as all powers of 6 leave a remainder of 6 when divided by 10. It follows that our answer is the remainder when 7^6 is divided by 11, or $7^6 \equiv \boxed{4} \pmod{11}$.

13. A positive integer is written on each corner of a square such that numbers on opposite vertices are relatively prime while numbers on adjacent vertices are not relatively prime. What is the smallest possible value of the sum of these 4 numbers?

Solution: Let the numbers be a, b, c, and d in order. To minimize the sum of the 4 numbers, let $\gcd(a, b) = 2$. It follows that c and d are not divisible by 3, so let $\gcd(c, d) = 3$. It follows that a and b are not divisible by 3, so let $\gcd(a, d) = 5$ and let $\gcd(b, c) = 7$. Clearly this configuration satisfies the given configuration, letting $a = 10$, $b = 14$, $c = 21$, and $d = 15$. We can convince ourselves that this configuration minimizes the sum of the four numbers, so $a + b + c + d = \boxed{60}$ is the minimum value.

14. Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Solution: Notice that when x is even and greater than 0, the given sum is divisible by 3, as each term after the first will leave a remainder of 1 when divided by 3 and $6 \cdot 1 \equiv 0 \pmod 3$. Notice that when x is odd, the given sum is divisible by 5 as the sum is equivalent to $(-2)^x + 0 + 2^x + 1^x + 3^x + (-3)^x + (-1)^x \equiv 0 \pmod 5$. It follows that there are $\boxed{0}$ positive values of x which make the sum prime.

15. Find the largest integer less than 2012 all of whose divisors have at most two 1s in their binary representations.

Solution: Notice that if the property holds for an even integer n, then the property also holds for the integer $\frac{n}{2}$, and vice versa. We can observe that $2012 = 11111011100_2$. The largest integer less than 2012 with at most two 1s in its binary representation is therefore $11000000000_2 = 1024+512 = 1536$. By the property stated above, this number will have the property if $\frac{1536}{512} = 3$ has the property, and we can easily observe that 3 has the property, so our answer is $\boxed{1536}$.

3 Sources

1. 2008 November Harvard MIT Math Tournament General Problem 6
2. 2008 November Harvard MIT Math Tournament General Problem 8
3. 2011 November Harvard MIT Math Tournament General Problem 4
4. 2012 November Harvard MIT Math Tournament General Problem 1
5. 2012 November Harvard MIT Math Tournament General Problem 7
6. 2013 November Harvard MIT Math Tournament General Problem 1
7. 2013 November Harvard MIT Math Tournament General Problem 6
8. 2013 November Harvard MIT Math Tournament General Problem 8
9. 2014 November Harvard MIT Math Tournament General Problem 3
10. 2016 November Harvard MIT Math Tournament General Problem 9
11. 2009 November Harvard MIT Math Tournament Theme Problem 9
12. 2012 November Harvard MIT Math Tournament Theme Problem 5
13. 2016 November Harvard MIT Math Tournament Theme Problem 4
14. 2011 November Harvard MIT Math Tournament Team Problem 1

15. 2012 November Harvard MIT Math Tournament Team Problem 3