# Number Theory Handout 3

Walker Kroubalkian

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## 1 Problems

**1.** We say $s$ grows to $r$ if there exists some integer $n > 0$ such that $s^n = r$. Call a real number $r$ sparse if there are only finitely many real numbers $s$ that grow to $r$. Find all real numbers that are sparse.

**2.** How many integers between 2 and 100 inclusive cannot be written as $m \cdot n$, where $m$ and $n$ have no common factors and neither $m$ nor $n$ is equal to 1? Note that there are 25 primes less than 100.

**3.** Determine the remainder when

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}$$

is divided by 7.

**4.** What is the sum of all of the distinct prime factors of $25^3 - 27^2$?

**5.** Find the number of ordered 2012-tuples of integers $(x_1, x_2, ..., x_{2012})$, with each integer between 0 and 2011 inclusive, such that the sum $x_1 + 2x_2 + 3x_3 + \cdots + 2012x_{2012}$ is divisible by 2012.

**6.** What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

**7.** Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

**8.** How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, ...$?

**9.** Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

**10.** Let the sequence $a_i$ be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \le n \le 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

**11.** Five guys each have a positive integer (the integers are not necessarily distinct). The greatest common divisor of any two guys numbers is always more than 1, but the greatest common divisor of all the numbers is 1. What is the minimum possible value of the product of the numbers?

**12.** Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^{4^{3^{2^{1}}}}}}$ repeatedly until a single base 12 digit remains. Find this digit.

**13.** A positive integer is written on each corner of a square such that numbers on opposite vertices are relatively prime while numbers on adjacent vertices are not relatively prime. What is the smallest possible value of the sum of these 4 numbers?

**14.** Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

**15.** Find the largest integer less than 2012 all of whose divisors have at most two 1s in their binary representations.

## 2 Sources

**1.** 2008 November Harvard MIT Math Tournament General Problem 6
**2.** 2008 November Harvard MIT Math Tournament General Problem 8
**3.** 2011 November Harvard MIT Math Tournament General Problem 4
**4.** 2012 November Harvard MIT Math Tournament General Problem 1
**5.** 2012 November Harvard MIT Math Tournament General Problem 7
**6.** 2013 November Harvard MIT Math Tournament General Problem 1
**7.** 2013 November Harvard MIT Math Tournament General Problem 6
**8.** 2013 November Harvard MIT Math Tournament General Problem 8
**9.** 2014 November Harvard MIT Math Tournament General Problem 3
**10.** 2016 November Harvard MIT Math Tournament General Problem 9
**11.** 2009 November Harvard MIT Math Tournament Theme Problem 9
**12.** 2012 November Harvard MIT Math Tournament Theme Problem 5
**13.** 2016 November Harvard MIT Math Tournament Theme Problem 4
**14.** 2011 November Harvard MIT Math Tournament Team Problem 1
**15.** 2012 November Harvard MIT Math Tournament Team Problem 3