Number Theory Handout 1 Answers and Solutions

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1 Answers

1. 12

2. 38

3. 23

4. 5

5. 9030

6. 5

7. 5

8. 7

9. 9

10. -125

11. $\{4, 9, 25\}$ or $\{2^2, 3^2, 5^2\}$

12. $-x^2 + 1$ or $1 - x^2$

13. 72381

14. 24

15. 39

2 Solutions

1. Compute the smallest positive integer with exactly six different factors.

Solution: We remember that if the prime factorization of a number is $p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_n^{e_n}$, the number of factors of the number is $(e_1 + 1) \cdot (e_2 + 1) \cdot (e_3 + 1) \cdots (e_n + 1)$. This fact will only be stated in this solution but is constantly used throughout this handout. Setting this equal to 6, we notice that there can be at most two factors inside the parentheses, as $6 = 2 \times 3$, and it is impossible to represent 6 as a product of more numbers that are greater than 1. It follows that our number must either be of the form $p_1^2 \cdot p_2^1$ as $(2 + 1) \cdot (1 + 1) = 6$ or of the form p_1^5 as $(5 + 1) = 6$. The smallest number of the form $p_1^2 \cdot p_2^1$ is $2^2 \cdot 3^1 = 12$ and the smallest number of the form p_1^5 is $2^5 = 32$. Therefore, our answer is $\boxed{12}$.

2. We define a positive integer p to be *almost prime* if it has exactly one divisor other than 1 and p. Compute the sum of the three smallest numbers which are *almost prime*.

Solution: Refer to the solution to problem 1. It follows that the number must be of the form p_1^2 as $(2+1) = 3$. Therefore, the three smallest numbers are $2^2 = 4$, $3^2 = 9$, and $5^2 = 25$. Therefore our answer is $4 + 9 + 25 = \boxed{38}$.

3. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \times 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

Solution: We will do casework on the value of $f(x)$:

Case 1: $f(x) = 1$

If $f(x) = 1$, it follows that x has no prime factors, or $x = 1 < 2$. Therefore, we have 0 numbers in this case.

Case 2: $f(x) = 2$

If $f(x) = 2$, it follows that x's only prime factor is 2. It follows that x must be a power of 2, leading to the powers from 2^1 to 2^6 for a total of 6 numbers in this case.

Case 3: $f(x) = 3$

If $f(x) = 3$, by similar logic to case 2, we must have x is a power of 3. This gives us the powers from 3^1 to 3^4, for a total of 4 numbers in this case.

Case 4: $f(x) = 4$

It is impossible for $f(x)$ to equal 4 as $4 = 2^2$ and 4 itself is not prime. By similar reasoning, it is impossible for $f(x)$ to equal 8 or 9 so we will exclude those cases.

Case 5: $f(x) = 5$

By similar logic to cases 2 and 3, x must be a power of 5. This gives us the powers from 5^1 to 5^2 for a total of 2 numbers in this case.

Case 6: $f(x) = 6$

This is the one truly annoying case. We must have x is of the form $2^{e_1} \cdot 3^{e_2}$ where both e_1 and e_2 are integers. By brute force we get the only possibilities are $2^1 3^1, 2^1 3^2, 2^1 3^3, 2^2 3^1, 2^2 3^2, 2^3 3^1, 2^3 3^2, 2^4 3^1$, and $2^5 3^1$. This gives us a total of 9 numbers in this case.

Case 7: $f(x) = 7$

By similar logic to cases $2, 3$, and 5, we have that x must be a power of 7. Thus, we have all of the powers from 7^1 to 7^2 for a total of 2 numbers in this case.

Adding up all of our cases, we have a total of $6+4+2+9+2 = \boxed{23}$ numbers with these properties.

4. For a positive integer a, let $f(a)$ be the average of all positive integers b such that $x^2+ax+b = 0$ has integer solutions. Compute the unique value of a such that $f(a) = a$.

Solution: We remember that if p and q are the solutions to the quadratic $x^2 + ax + b = 0$, then by Vieta's Formulas, $-a = p + q$ and $b = pq$. Because a is positive, both p and q must be negative for p to be a positive integer. Therefore, we wish to compute

$$\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} (-i) \times (-a + i) = \sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} i \times (a - i) = a \times \sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} i - \sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} i^2$$

We remember by Gauss Sums that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. It follows that if we let $\frac{a}{2} = z$, then our expression is equivalent to

$$\frac{a \cdot z \cdot (z+1)}{2} - \frac{z \cdot (z+1) \cdot (2z+1)}{6} = \frac{3az^2 + 3az - 2z^3 - 3z^2 - z}{6}$$

. We wish to determine when this is equal to $a \cdot z$. There are two possibilities. Either a is even, in which case, $z = \frac{a}{2}$, or a is odd, in which case $z = \frac{a-1}{2}$. First, assume a is even. It follows that

$$\frac{a^2}{2} = \frac{2a^3 + 3a^2 - 2a}{24} \to 2a^2 - 9a - 2 = 0$$

which has no integer solutions for a. It follows that a must be odd. In this case, we get

$$\frac{a(a-1)}{2} = \frac{3a(a-1)^2 + 6a(a-1) - (a-1)^3 - 3(a-1)^2 - 2(a-1)}{24} \to a^2 - 6a + 5 = 0$$

It follows that $a = \boxed{5}$ as desired.

5. What is the smallest number over 9000 that is divisible by the first four primes?

Solution: The first four primes are $2, 3, 5$, and 7. It follows that any number which is divisible by all of them must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 = 210$. The smallest multiple of 210 which is greater than 9000 is $210 \times \lceil \frac{9000}{210} \rceil = \boxed{9030}$ as desired.

6. Consider a sequence given by $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$, where $a_0 = a_1 = a_2 = 1$. What is the remainder when a_{2013} is divided by 7?

Solution: We wish to find a pattern in the remainder when a_n is divided by 7. By using the recurrence equation, we can calculate the first few values of a_n. However, by looking at the recurrence $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$, we find that the only property that effects the remainder when a_n is divided by 7 is the remainders when a_{n-1}, a_{n-2}, and a_{n-3} are divided by 7. Therefore, it suffices to just calculate the remainders when values of a_n are divided by 7 until we get a triple that we have already seen. This gives us the sequence

$$1, 1, 1, 5, 2, 4, 1, 1, 1, \cdots$$

as we can see, the sequence repeats every 6 values, so we get that

$$a_{2013} \pmod 7 = a_{2013 \pmod 6} \pmod 7 = a_3 = \boxed{5}$$

as desired.

7. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and *boring*?

Solution: Notice that the number $\overline{nn\cdots n}$ where each n is a digit is equal to $n \times \overline{11\cdots 1}$ where the number of 1's and n's are equal. It follows that only numbers with a repeating digit of 1 could possibly be prime if the number has more than 1 digit. These numbers are $11, 111$, and 1111. By brute force, we can find that 11 is the only number in this list that is prime. However, if the number only has 1 digit, it does not matter which digit repeats. This adds the four primes $2, 3, 5$, and 7. Thus, there are $\boxed{5}$ positive integers less than 10000 which are both prime and *boring*.

8. Given a number n in base 10, let $g(n)$ be the base-3 representation of n. Let $f(n)$ be equal to the base-10 number obtained by interpreting $g(n)$ in base 10. Compute the smallest positive integer $k \geq 3$ that divides $f(k)$.

Solution: Let the base-3 representation of n be $\overline{a_x a_{x-1} \cdots a_0}$, where all of the a_i are digits from the set $\{0, 1, 2\}$. It follows that

$$n = 3^x \times a_x + 3^{x-1} \times a_{x-1} + \cdots + 3^0 \times a_0$$

When this is interpreted in base 10, we will get the value

$$f(n) = 10^x \times a_x + 10^{x-1} \times a_{x-1} + \cdots 10^0 \times a_0$$

With this understanding of bases, we get the following values for $f(n)$ for small values of n: $f(3) = 10, f(4) = 11, f(5) = 12, f(6) = 20, f(7) = 21$. Because $\frac{21}{7}$ is an integer, we have that our answer is $\boxed{7}$.

9. Given a 1962-digit number that is divisible by 9, let x be the sum of its digits. Let the sum of the digits of x be y. Let the sum of the digits of y be z. Compute the maximum possible value of z.

Solution: We notice that when we take the sum of the digits of a multiple of 9, we get a multiple of 9. Thus, x is a multiple of 9 which is between 1 (in the case that the 1962-digit number is $10000 \cdots 00$) and $9 \times 1962 = 17658$ (in the case that the 1962-digit number is $999 \cdots 99$). It follows that y is a multiple of 9 between 1 and $9 \times 4 = 36$ (in the case that $x = 9999$). It follows that z is a multiple of 9 between 1 and $2 + 9 = 11$ (in the case that $y = 29$). Therefore, no matter what the 1962-digit number was, $z = \boxed{9}$.

10. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are nonnegative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

Solution: We remember by Vieta's formulas that if a cubic formula of the form $x^3 + ax^2 + bx + c$ has roots r_1, r_2, and r_3, then $-a = r_1 + r_2 + r_3$, $b = r_1r_2 + r_1r_3 + r_2r_3$, and $-c = r_1r_2r_3$. The cubic could also be written in the form $(x - r_1) \times (x - r_2) \times (x - r_3)$. Because $f(0) = -64$, we must have $c = -64$, meaning $r_1r_2r_3 = 64$. Given this, we wish to maximize the value of $(-1 - r_1)(-1 - r_2)(-1 - r_3) = -(1 + r_1)(1 + r_2)(1 + r_3)$, meaning that we want to minimize the value of $(1 + r_1)(1 + r_2)(1 + r_3)$ when $r_1r_2r_3 = 64$. By experimentation, we can find that this is minimized when $r_1 = r_2 = r_3 = 4$, giving a value of $(1 + 4)^3 = 125$. For a more rigorous proof, we can use Holder's Inequality to find that

$$(1 + r_1)^{\frac{1}{3}} \times (1 + r_2)^{\frac{1}{3}} \times (1 + r_3)^{\frac{1}{3}} \geq 1 + \sqrt[3]{r_1r_2r_3} = 5$$

Cubing both sides immediately gives the result. However, we must remember that this is equal to $-f(-1)$. Therefore, the maximum value of $f(-1)$ is $\boxed{-125}$ as desired.

11. Find all square numbers which can be represented in the form $2^a + 3^b$, where a, b are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

Solution: First, assume b is 0. It follows that we wish to find numbers x and a such that $x^2 = 2^a + 1$. This rearranges to $(x - 1)(x + 1) = 2^a$. Because x is an integer, this is only possible if both $x - 1$ and $x + 1$ are powers of 2. If $x - 1$ is odd, it must be 1, so $x + 1$ is 3, leading to no solutions. If $x - 1$ is even, than one of $x - 1$ and $x + 1$ must be a multiple of 4 while the other is only a multiple of 2. This only leads to the solution of $x = 3$, or $x^2 = 9$, as this is the only value for which both $x - 1$ and $x + 1$ are powers of 2. Now, assume b is positive. It follows by the fact that perfect squares can only be equivalent to 0 or 1 $\pmod 3$ that a must be even. Now let $a = 2y$. It follows that $x^2 = 2^{2y} + 3^b$. Rearranging, we get $(x - 2^y)(x + 2^y) = 3^b$. Because at most one of $(x - 2^y)$ or

$(x+2^y)$ can be divisible by 3, this can only be possible if $x-2^y=1$ and $x+2^y=3^b$. It follows that $3^b-2^{y+1}=1$. However, we have been told that $3^x-2^y=1$ has no integer solutions when $x\geq 3$. Therefore, Plugging values in, we get the only pairs $(b,y+1)$ which work are $(2,3)$ and $(1,1)$. These pairs lead to the solutions $2^4+3^2=25$ and $2^0+3^1=4$, respectively. Therefore our solutions are $\boxed{\{4,9,25\}}$ or $\boxed{\{2^2,3^2,5^2\}}$ as desired.

12. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1,0,1\}$ and with least possible degree such that $P(2010)\equiv 1 \pmod 3$, $P(2011)\equiv 0 \pmod 3$, and $P(2012)\equiv 0 \pmod 3$.

Solution: Notice that if the coefficients are integers, than $P(2010)\equiv 1 \pmod 3$ implies $P(2010 \pmod 3)) = P(0)\equiv 1 \pmod 3$. Similarly, $P(1)\equiv 0 \pmod 3$ and $P(2)\equiv 0 \pmod 3$. Because $P(0)\equiv 1 \pmod 3$, we must have the constant term is 1. Notice that if the polynomial were linear, than if the coefficient of the x-term were a, we would have $a+1\equiv 0 \pmod 3$ and $2a+1\equiv 0 \pmod 3$ which is clearly impossible. If the polynomial were quadratic and of the form ax^2+bx+1, we would have $a+b\equiv 2 \pmod 3$ and $4a+2b\equiv 2 \pmod 3$. It follows that $b\equiv 0 \pmod 3$ and $a\equiv 2 \pmod 3$. Thus, the unique polynomial of minimum degree is $P(x)=\boxed{-x^2+1}$.

13. Compute the sum of all n for which the equation $2x+3y=n$ has exactly 2011 nonnegative $(x,y\geq 0)$ integer solutions.

Solution: Consider the nonnegative integer solution (x_m,y_m) with the maximum value of y. Notice that to get other solutions, we must decrease y_m by multiples of 2 and increase x_m by corresponding multiples of 3. It follows that the total number of solutions is $\lfloor\frac{y_m+2}{2}\rfloor$ due to the fact that we must include 0 as a possible value of y in the equation. Setting this equal to 2011 and solving gives us $y=4020$ and $y=4021$. Notice that when y is maximized, x must be less than or equal to 3, as otherwise we could decrease x by 3 to get a solution with a larger value of y. It follows that x is among the set $\{0,1,2\}$. Therefore, (x_m,y_m) is among the set $\{(0,4020),(1,4020),(2,4020),(0,4021),(1,4021),(2,4021)\}$. These give us values of n of $\{12060,12062,12064,12063,12065,12067\}$, respectively. Adding these up, we get a total of $\boxed{72381}$ as desired.

14. Find the largest integer that divides p^2-1 for all primes $p>3$.

Solution: We claim that the largest integer that divides p^2-1 for all primes $p>3$ is 24. Notice that $5^2-1=24$, so clearly the largest integer must be a factor of 24. Notice that all primes $p>3$ are not multiples of 3. Therefore, we either have $p\equiv 1 \pmod 3$ or $p\equiv 2 \pmod 3$. Squaring both gives us $p^2\equiv 1 \pmod 3$ or $p^2\equiv 1 \pmod 3$. Therefore, no matter what prime is chosen, $p^2-1\equiv 0 \pmod 3$, and therefore 3 must be a factor of p^2-1. We can also write p^2-1 as $(p-1)\times(p+1)$. Notice that when $p>3$, we must have p is odd, so both $(p-1)$ and $(p+1)$ are even. However, because they are consecutive even numbers, one must also be a multiple of 4. Therefore, 8 divides p^2-1. Thus, we have shown $3\cdot 8=\boxed{24}$ divides p^2-1 for all primes $p>3$, and we have also shown this is the largest integer with this property.

15. A positive integer $b\geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0<x<b$ and $0<y<b$) such that the number $x.y$ base b (that is, $x+\frac{y}{b}$) is an integer multiple of $\frac{x}{y}$. Find the number of *neat* integers less than or equal to 100.

Solution: Notice that we can write $x.y$ in base b as $x+\frac{y}{b}=\frac{xb+y}{b}$. If this is an integer multiple of $\frac{x}{y}$, we must have $\frac{\frac{xb+y}{b}}{\frac{x}{y}}=\frac{xby+y^2}{xb}=y+\frac{y^2}{xb}$ is an integer. It follows that y^2 is divisible by xb. Clearly, if this is true when $x>1$, than it will also be true when $x=1$. Therefore, b is *neat* if and only if we can find a number y less than b such that y^2 is divisible by b. Notice that if b has any prime factor which is repeated in its prime factorization, than we can divide b by that prime to get a number y which satisfies this property, as the result will clearly satisfy $y < b$ and it will satisfy y^2 is divisible by all other prime power factors of b. If we let the number of occurrences of the prime we chose be x, than the number of times it appears in the factorization of y^2 is $2x-2$, so we must have $2x - 2 \geq x$. However, this is always true when $x > 1$, so we have shown this value of y will satisfy the property. Therefore, any number which is not squarefree will be *neat*.

We will do casework on which prime factors occur more than once in the prime factorization of x.

Case 1: 2 occurs more than once.

If 2 occurs more than once, the number must be a multiple of 4. There are $\frac{100}{4} = 25$ multiples of 4 less than 100.

Case 2: 3 occurs more than once.

If 3 occurs more than once, the number must be a multiple of 9. There are $\lfloor \frac{100}{9} \rfloor = 11$ multiples of 9, but we have already counted multiples of 36, so we must subtract $\lfloor \frac{100}{36} \rfloor = 2$ for a total of 9 new multiples of 9 less than 100.

Case 3: 5 occurs more than once.

If 5 occurs more than once, the number must be a multiple of 25. There are $\frac{100}{25} = 4$ multiples of 25 less than or equal to 100 but we have already counted multiples of 100 so we must subtract $\frac{100}{100} = 1$ for a total of 3 new multiples of 25 less than 100.

Case 4: 7 occurs more than once.

If 7 occurs more than once, the number must be a multiple of 49. There are $\lfloor \frac{100}{49} \rfloor = 2$ multiples of 49 less than or equal to 100.

Adding up all of our cases, we have a total of $25 + 9 + 3 + 2 = \boxed{39}$ *neat* numbers less than or equal to 100.

3 Sources

1. 2014 Stanford Math Tournament General Problem 9
2. 2014 Stanford Math Tournament General Problem 13
3. 2014 Stanford Math Tournament General Problem 22
4. 2014 Stanford Math Tournament General Problem 23
5. 2013 Stanford Math Tournament General Problem 4
6. 2013 Stanford Math Tournament General Problem 10
7. 2012 Stanford Math Tournament General Problem 13
8. 2012 Stanford Math Tournament General Problem 14
9. 2012 Stanford Math Tournament General Problem 16
10. 2012 Stanford Math Tournament General Problem 19
11. 2011 Stanford Math Tournament General Problem 6
12. 2011 Stanford Math Tournament General Problem 11
13. 2011 Stanford Math Tournament General Problem 22
14. 2014 Stanford Math Tournament Advanced Topics Problem 2
15. 2013 Stanford Math Tournament Advanced Topics Problem 6