Number Theory Handout 1

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1 Problems

1. Compute the smallest positive integer with exactly six different factors.

2. We define a positive integer p to be *almost prime* if it has exactly one divisor other than 1 and p. Compute the sum of the three smallest numbers which are *almost prime*.

3. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \times 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

4. For a positive integer a, let $f(a)$ be the average of all positive integers b such that $x^2 + ax + b = 0$ has integer solutions. Compute the unique value of a such that $f(a) = a$.

5. What is the smallest number over 9000 that is divisible by the first four primes?

6. Consider a sequence given by $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$, where $a_0 = a_1 = a_2 = 1$. What is the remainder when a_{2013} is divided by 7?

7. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and *boring*?

8. Given a number n in base 10, let $g(n)$ be the base-3 representation of n. Let $f(n)$ be equal to the base-10 number obtained by interpreting $g(n)$ in base 10. Compute the smallest positive integer $k \geq 3$ that divides $f(k)$.

9. Given a 1962-digit number that is divisible by 9, let x be the sum of its digits. Let the sum of the digits of x be y. Let the sum of the digits of y be z. Compute the maximum possible value of z.

10. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are nonnegative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

11. Find all square numbers which can be represented in the form $2^a + 3^b$, where a, b are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

12. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

13. Compute the sum of all n for which the equation $2x + 3y = n$ has exactly 2011 nonnegative $(x, y \geq 0)$ integer solutions.

14. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

15. A positive integer $b \geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0 < x < b$ and $0 < y < b$) such that the number $x.y$ base b (that is, $x + \frac{y}{b}$) is an integer multiple of $\frac{x}{y}$. Find the number of *neat* integers less than or equal to 100.

2 Sources

1. 2014 Stanford Math Tournament General Problem 9
2. 2014 Stanford Math Tournament General Problem 13
3. 2014 Stanford Math Tournament General Problem 22
4. 2014 Stanford Math Tournament General Problem 23
5. 2013 Stanford Math Tournament General Problem 4
6. 2013 Stanford Math Tournament General Problem 10
7. 2012 Stanford Math Tournament General Problem 13
8. 2012 Stanford Math Tournament General Problem 14
9. 2012 Stanford Math Tournament General Problem 16
10. 2012 Stanford Math Tournament General Problem 19
11. 2011 Stanford Math Tournament General Problem 6
12. 2011 Stanford Math Tournament General Problem 11
13. 2011 Stanford Math Tournament General Problem 22
14. 2014 Stanford Math Tournament Advanced Topics Problem 2
15. 2013 Stanford Math Tournament Advanced Topics Problem 6