Number Theory Handout #4 Answers and Solutions

Walker Kroubalkian

December 26, 2017

1 Answers

1. 601

2. 29016

3. 781

4. 2005

5. 2^{813}

6. 238

7. 75

8. $31, 15121$

9. 50

10. $(34, 55)$

11. 2350

12. $1, 2, 3$

13. 46

14. 2460

15. 1260

2 Solutions

1. The **three-digit** prime number p is written in base 2 as p_2 and in base 5 as p_5, and the two representations share the same last 2 digits. If the ratio of the number of digits in p_2 to the number of digits in p_5 is 5 to 2, find all possible values of p.

Solution: Notice that the largest number in base 2 with 5 digits is $11111_2 = 31$ and the smallest number in base 2 with 15 digits is $100000000000000_2 = 16384$. It follows that our number must have 10 digits in base 2 and 4 digits in base 5, which means that if the number is n, then $512 \leq n < 625$. Now we will do casework on the last two digits of n in each of the respective bases. If the last two digits are 00 in each base, then n must be divisible by 25 and 4. The only number which satisfies this condition is $n = 600$ which is not prime. If the last two digits of n are 10, then $n \equiv 5 \pmod{25}$ and $n \equiv 2 \pmod{4}$. The only number which satisfies these conditions is $n = 630$ which is not prime. If the last two digits of n are 01, then $n \equiv 1 \pmod{100}$, and the only number which satisfies this condition is $n = 601$ which is prime. If the last two digits of n are 11, then $n \equiv 31$

(mod 100). The only number which satisfies this condition is $n = 631$ which is prime, but greater than 625. It follows that our answer is $\boxed{601}$.

2. Find the sum of all possible n such that n is a positive integer and there exist a, b, c real numbers such that for every integer m, the quantity $\frac{2013m^3+am^2+bm+c}{n}$ is an integer.

Solution: If we let $a = dn$, $b = en$, and $c = fn$, then it follows that we want $\frac{2013}{n}m^3+dm^2+em+f$ to be an integer. However, we must also have $\frac{2013}{n}(m+1)^3 + d(m+1)^2 + e(m+1) + f$ is an integer, and therefore their difference must also be an integer. It follows that $\frac{6039}{n}m^2 + gm + h$ must be an integer for arbitrary real numbers g and h. However, we must also have that $\frac{6039}{n}(m+1)^2 + g(m+1) + h$ is an integer, and therefore their difference, $\frac{12078}{n}m + i$ must be an integer for some real number i. It follows that $\frac{12078}{n}(m+1) + i$ is an integer, and therefore we need $\frac{12078}{n}$ to be an integer. $12078 = 2 \cdot 3^2 \cdot 11^1 \cdot 61^1$, so it follows that the sum of the values of n which work is $(1+2)(1+3+9)(1+11)(1+61) = \boxed{29016}$ as desired.

3. A number is between 500 and 1000 and has a remainder of 6 when divided by 25 and a remainder of 7 when divided by 9. Find the only odd number to satisfy these requirements.

Solution: Let the number be n. By the Chinese Remainder Theorem, we can find a unique residue for $n \pmod{9 \cdot 25}$. The only number which is less than 225 with these properties is 106. It follows that $n \equiv 106 \pmod{225}$, and the only odd number in the given range with this property is $n = 675 + 106 = \boxed{781}$

4. Given $f_1 = 2x - 2$ and $k \geq 2$, define $f_k(x) = f_1(f_{k-1}(x))$ to be a real-valued function of x. Find the remainder when $f_{2013}(2012)$ is divided by the prime 2011.

Solution: We can rewrite $f_1(x)$ as $f_1(x) = 2(x-1)$. It follows that we wish to investigate $2(2(2(\cdots 2(2012-1)-1\cdots)-1)-1)$. Expanding this, we get that we wish to calculate $2^{2013} \cdot 2012 - 2^{2013} - 2^{2012} - 2^{2011} - \cdots - 2^1 = 2012 \cdot 2^{2013} - (2^{2014} - 2) = 2010 \cdot 2^{2013} + 2$. We know that $2^{2010} \equiv 1 \pmod{2011}$, so it follows that $2010 \cdot 2^{2013} + 2 \equiv 2010 \cdot 8 + 2 \equiv -6 \equiv \boxed{2005} \pmod{2011}$ as desired.

5. Consider the roots of the polynomial $x^{2013} - 2^{2013} = 0$. Some of these roots also satisfy $x^k - 2^k = 0$, for some integer $k < 2013$. What is the product of this subset of roots?

Solution: Let $p(x)$ be a function which returns the smallest value of k such that $x^k = 2^k$ where x is a root of 2. It follows that the number of roots which satisfy $p(x) = n$ is $\phi(n)$. We also know that if x is a root, then there exists another root y which can be uniquely paired with x such that $xy = 4$. It follows that the product of all of the roots x such that $x^n = 2^n$ and n is the smallest exponent with this property is $4^{\frac{\phi(n)}{2}} = 2^{\phi(n)}$. It follows that our answer is

$$\prod_{n|2013, p\neq 2013} 2^{\phi(n)} = 2^(\sum_{n|2013, n\neq 2013} \phi(n)) = 2^{2013-\phi(2013)} = \boxed{2^{813}}$$

6. Denote by $S(a, b)$ the set of integers k that can be represented as $k = a \cdot m + b \cdot n$, for some non-negative integers m and n. So, for example, $S(2, 4) = \{0, 2, 4, 6, ...\}$. Then, find the sum of all possible positive integer values of x such that $S(18, 32)$ is a subset of $S(3, x)$.

Solution: Notice that x must be relatively prime to 3 as otherwise the value 32 would not be in $S(3, x)$ despite being in $S(18, 32)$. Also, by the Chicken McNugget Theorem, every value which is greater than $mn - m - n$ will be in $S(m, n)$. It follows that we need to find the largest value x such that $\mid S(3, x) \mid = 3x - 3 - x = 2x - 3 < 32$. Clearly, the largest value is $x = 17$. Therefore, we need to find the sum of the positive integers n which are less than 18 and not divisible by 3. The sum of these numbers is $1+2+4+5+7+8+10+11+13+14+16+17 = 3+9+15+21+27+33 = 108$.

However, we can notice that if $x = 20, 23, 26, 29, 32$, then 32 can still be achieved. It follows that our answer is $108 + 20 + 23 + 26 + 29 + 32 = \boxed{238}$.

7. Let σ_n be a permutation of $\{1, ..., n\}$; that is, $\sigma_n(i)$ is a bijective function from $\{1, ..., n\}$ to itself. Define $f(\sigma)$ to be the number of times we need to apply σ to the identity in order to get the identity back. For example, f of the identity is just 1, and all other permutations have $f(\sigma) > 1$. What is the smallest n such that there exists a σ_n with $f(\sigma_n) = 2013$?

Solution: Define a cycle of a permutation $\{1, ..., n\}$ to be a subset of elements of the identity $\{c_1, c_2, ..., c_x\}$ such that $\sigma(c_1) = c_2$, $\sigma(c_2) = c_3$, and so on, where $\sigma(c_x) = c_1$. It follows that $f(\sigma)$ is the least common multiple of the lengths of all of the cycles in σ. Therefore, we wish to find the smallest sum of a group of numbers which have a least common multiple of $2013 = 3^1 \cdot 11^1 \cdot 61^1$. Clearly, the smallest sum is $3 + 11 + 61 = \boxed{75}$.

8. Given that $468751 = 5^8 + 5^7 + 1$ is a product of two primes, find both of them.

Solution: Notice that $5^8+5^7+1 = (5^3-1)5^5+(5^3-1)5^4+5^5+5^4+1$. This can further be reduced to $(5^3-1)5^5+(5^3-1)5^4+(5^3-1)5^2+(5^3-1)5^1+5^2+5^1+1$. Because $5^3-1 = (5-1)(5^2+5+1)$, it follows that the original sum is a multiple of $5^2 + 5 + 1 = 31$. It follows that the prime factors are 31 and 15121. Therefore our answer is $\boxed{31, 15121}$

9. How many zeros does the product of the positive factors of 10000 (including 1 and 10000) have?

Solution: Notice that $10000 = 2^4 \cdot 5^4$ has 25 factors. Because each factor x can be paired with the unique factor $\frac{10000}{x}$, it follows that the product of the positive factors is $10000^{\frac{25}{2}} = 10^{50}$. It follows that our answer is $\boxed{50}$.

10. Find the ordered pair of positive integers (x, y) such that $144x - 89y = 1$ and x is minimal.

Solution: We wish to find the value of $144^{-1} \equiv 55^{-1} \pmod{89}$. It follows that we wish to solve the equation $55x = 89z + 1$. Taking this equation $\pmod{11}$ gives us $z + 1 \equiv 0 \pmod{11}$. Taking the equation $\pmod 5$ gives us $4z + 1 \equiv 0 \pmod 5$. It follows that $z \equiv 10 \pmod{11}$ and $z \equiv 1 \pmod 5$. It follows that $z \equiv 21 \pmod{55}$, and therefore $55x = 89 \cdot 21 + 1 = 1870$. It follows that the smallest value of x is $\frac{1870}{55} = 34$. This value of x corresponds to the ordered pair $\boxed{(34, 55)}$.

11. Let m and n be integers such that $m + n$ and $m - n$ are prime numbers less than 100. Find the maximal possible value of mn.

Solution: Notice that to maximize mn, we wish to minimize $m-n$ and we wish to maximize $m+n$ as $4mn = (m + n)^2 - (m - n)^2$. It follows that this will occur when $m - n = 3$ and $m + n = 97$. Therefore, $m = 50$ and $n = 47$, and it follows that the maximum value of mn is $47 \cdot 50 = \boxed{2350}$.

12. For a positive integer n, let $\phi(n)$ denote the number of positive integers between 1 and n, inclusive, which are relatively prime to n. We say that a positive integer k is total if $k = \frac{n}{\phi(n)}$, for some positive integer n. Find all total numbers.

Solution: We know that $\frac{n}{\phi(n)} = \prod_{p|n} \frac{p}{p-1}$. Listing $\frac{p}{p-1}$ for small values of p gives us 2, $\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{6}$, $\frac{11}{10}$. Clearly, at most one factor of 2 can appear in the numerator of the product of these numbers, so primes which are $1 \pmod 4$ cannot be divisors of n due to the multiplicative nature of $\frac{n}{\phi(n)}$. It follows that the only values of n which work are numbers of the forms $1, 2^x$, and $2^x 3^y$. These forms correspond to the total numbers 1, 2, and 3. It follows that our answer is $\boxed{1, 2, 3}$

13. Suppose that positive integers $a_1, a_2, ..., a_{2014}$ (not necessarily distinct) satisfy the condition that: $\frac{a_1}{a_2}, \frac{a_2}{a_3}, ..., \frac{a_{2013}}{a_{2014}}$ are pairwise distinct. What is the minimal possible number of distinct numbers in $\{a_1, a_2, ..., a_{2014}\}$?

Solution: Notice that if a set has n distinct positive integer elements, the total number of distinct fractions of the form $\frac{a}{b}$ where a and b are elements in the set is $n(n-1)+1$. The smallest value of n such that $n^2-n+1 \geq 2013$ is $n = 46$ as $46 \cdot 45 + 1 = 2071 > 2013$ while $45 \cdot 44 + 1 = 1980 < 2013$. It follows that our answer is $\boxed{46}$.

14. A unitary divisor d of a number n is a divisor n that has the property $\gcd(d, \frac{n}{d}) = 1$. If $n = 1620$, what is the sum of all unitary divisors of d?

Solution: Notice that if d is a unitary divisor of n, then if p is a prime factor of d, the largest power of p which divides n must also divide d, as otherwise $\frac{n}{d}$ would be divisible by p. It follows that finding unitary divisors is equivalent to dividing the largest prime power factors of n. $1620 = 2^2 \cdot 3^4 \cdot 5$. It follows that the unitary divisors of 1620 are $1, 4, 5, 20, 81, 324, 405, 1620$. Adding these up, we find that the sum of the unitary divisors of 1620 is $\boxed{2460}$.

Note: The official answer key for this question said the answer was 2400. However, because an official solution was never provided, I believe the answer presented above is actually the correct answer.

15. Let a, b, c be positive integers such that $\gcd(a, b) = 2$, $\gcd(b, c) = 3$, $\text{lcm}(a, c) = 42$, and $\text{lcm}(a, b) = 30$. Find abc.

Solution: Remembering that $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$, we can find that $ab = 60$. Because $3 \mid c$, $2 \nmid c$, and $c \mid 42$ while $a \mid 30$, we must have $7 \mid c$. It follows that $c = 3 \cdot 7 = 21$. Therefore our answer is $60 \cdot 21 = \boxed{1260}$.

3 Sources

1. 2013 Berkeley Math Tournament Spring Individual Problem 8
2. 2013 Berkeley Math Tournament Spring Individual Problem 16
3. 2013 Berkeley Math Tournament Spring Discrete Problem 1
4. 2013 Berkeley Math Tournament Spring Discrete Problem 4
5. 2013 Berkeley Math Tournament Spring Discrete Problem 5
6. 2013 Berkeley Math Tournament Spring Discrete Problem 7
7. 2013 Berkeley Math Tournament Spring Discrete Problem 10
8. 2014 Berkeley Math Tournament Fall Team Problem 18
9. 2014 Berkeley Math Tournament Fall Team Problem 10
10. 2014 Berkeley Math Tournament Fall Team Problem 9
11. 2014 Berkeley Math Tournament Spring Individual Problem 6
12. 2014 Berkeley Math Tournament Spring Discrete Problem 7
13. 2014 Berkeley Math Tournament Spring Discrete Problem 8
14. 2014 Berkeley Math Tournament Spring Team Problem 10
15. 2015 Berkeley Math Tournament Fall Individual Problem 12