Number Theory Handout #5 Answers and Solutions

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1 Answers

1. $4,060,225$

2. 4

3. 10

4. $2,730$

5. $(24,70)$

6. 1111011111000

7. 408

8. 48

9. 23

10. $1,012,032$

11. 315

12. 25

13. π

14. 20

15. 42

2 Solutions

1. An integer-valued function f satisfies $f(2) = 4$ and $f(mn) = f(m)f(n)$ for all integers m and n. If f is an increasing function, determine $f(2015)$.

Solution: Notice that $f(2) = f(1)f(2)$. It follows by $f(2) \neq 0$ that $f(1) = 1$. From here we can notice that if $f(n) = n \cdot |n|$, then all of the conditions are satisfied. It follows that $f(2015) = 2015^2 = \boxed{4,060,225}$.

2. The number 2^{29} has a 9-digit decimal representation that contains all but one of the 10 (decimal) digits. Determine which digit is missing.

Solution: Remembering that the sum of the digits of a number leaves the same remainder as the number itself when divided by 9, we try taking the remainder when 2^{29} is divided by 9. By Euler's Theorem, this is equivalent to $2^{29} \equiv 2^5 \equiv 5 \pmod 9$. Now notice that if all of the 10 digits were included, they would sum to 36 which is equivalent to $0 \pmod 9$. It follows that the missing digit is equivalent to $0 - 5 \equiv 4 \pmod 9$, and therefore the missing digit is $\boxed{4}$.

3. Determine the largest integer n such that 2^n divides the decimal representation given by some permutation of the digits $2, 0, 1$, and 5. (For example, 2^1 divides 2150. It may start with 0.)

Solution: Notice that $n = 10$ can be achieved with the number $5120 = 2^{10} \cdot 5^1$. It suffices to show that it is impossible to achieve $n = 11$. The only 4-digit multiples of 2^{11} are $2048, 4096, 6144$, and 8192. Therefore, our answer is $n = \boxed{10}$.

4. Determine the greatest integer N such that N is a divisor of $n^{13} - n$ for all integers n.

Solution: Notice that $2^{13}-2 = 8190 = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1$ and $3^{13}-3 = 1594320 = 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1 \cdot 73^1$. It follows that N is a divisor of $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1$. Because n and n^{13} have the same parity, we know N is divisible by 2. We can factor the given expression as

$$n^{13} - n = n(n-1)(n^2+n+1)(n+1)(n^2-n+1)(n^2+1)(n^4-n^2+1)$$

Because $n-1$, n, and $n+1$ are all factors of this expression, we know 3^1 divides N as one of these three factors has to be divisible by 3^1. By Fermat's Little Theorem, one of n or $n^{12}-1$ has to be divisible by 5, one of n or $n^{12}-1$ has to be divisible by 7, and one of n or $n^{12}-1$ has to be divisible by 13. It follows that our answer is $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1 = \boxed{2,730}$.

5. There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

Solution: The left hand side of this equation is equivalent to $\frac{k(k+1)(2k+1)}{6}$. If we let $k = 6y$, then this is equivalent to $y(6y+1)(12y+1)$. We can notice that when $y = 4$, this is equivalent to $4 \cdot 25 \cdot 49 = (2 \cdot 5 \cdot 7)^2 = 70^2$. Therefore, our pair is $(k, n) = (6 \cdot 4, 70) = \boxed{(24, 70)}$.

6. Determine the smallest positive integer containing only 0 and 1 as digits that is divisible by each integer 1 through 9.

Solution: In order for the number to be divisible by 8, we must have that the last three digits are 0. This will also guarantee that the number is divisible by $1, 2, 4$, and 5. In order for the number to be divisible by 9, we must have that the sum of its digits is at least 9. Therefore, if the number has 9 1's and some number of 0's, then we can also guarantee that it is divisible by $3, 6$, and 9. The hardest part is finding what numbers will be divisible by 7. Through trial and error with the fact that $10^1 \equiv 3 \pmod 7$, $10^2 \equiv 2 \pmod 7$, $10^3 \equiv 6 \pmod 7$, $10^4 \equiv 4 \pmod 7$, $10^5 \equiv 5 \pmod 7$, and $10^6 \equiv 1 \pmod 7$, we can find that the smallest number which satisfies all of these conditions is $\boxed{1111011111000}$.

7. There exist right triangles with integer side lengths such that the legs differ by 1. For example, $3-4-5$ and $20-21-29$ are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

Solution: Let the smaller leg of such a right triangle be n, and let its hypotenuse be y. It follows that $2n^2 + 2n + 1 - y^2 = 0$. Using the quadratic formula on this expression in terms of n, we get $n = \dfrac{-2 \pm \sqrt{8y^2 - 4}}{4}$. It follows that $8y^2 - 4 = x^2$, for some value of x. Because the left hand side is divisible by 4, it follows that x is even and that $2y^2 - 1 = z^2$ for some integer z such that $n = \frac{z-1}{2}$. Rearranging this equation, we get the common Pell Equation $z^2 - 2y^2 = -1$. Through trial and error, we can find that the first few pairs of integers (z, y) which satisfy this equation are $(z, y) = (1, 1), (7, 5)$, and $(41, 29)$. Using the common fact that solutions to Pell equations can be recursively used to generate other solutions, we can find that if (z, y) is a solution, $(3z+4y, 2z+3y)$

is also a solution. It follows that the next solution is $(z, y) = (239, 169)$. Working backwards, this produces the right triangle with side lengths 119, 120, and 169. The perimeter of this right triangle is $119 + 120 + 169 = \boxed{408}$.

8. When 20^{16} is divided by 16^{20} and expressed in decimal form, what is the number of digits to the right of the decimal point? Trailing zeroes should not be included.

Solution: $20^{16} = 2^{32} \cdot 5^{16}$ while $16^{20} = 2^{80}$. It follows that the given quotient will give the fraction $\frac{5^{16}}{2^{48}} = (\frac{5}{8})^{16} = (0.625)^{16}$. Each time 0.625 is multiplied by itself, it will add exactly 3 digits after the decimal point, so in total, the quotient will have $3 \cdot 16 = \boxed{48}$ digits to the right of the decimal point.

9. Let $2016 = a_1 \times a_2 \times \cdots \times a_n$ for some positive integers $a_1, a_2, ..., a_n$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_n$.

Solution: Notice that $2016 = 2^5 \cdot 3^2 \cdot 7^1$. Because $x + y - xy = 1 - (x - 1)(y - 1)$, to minimize the value of $a_1 + \cdots + a_n$, we want to make each of the values from a_1 to a_n as small as possible while they are still greater than 1. It follows that the sum will be minimized when the values from a_1 to a_n are the prime factors of 2016. It follows that our answer is $2 + 2 + 2 + 2 + 2 + 3 + 3 + 7 = \boxed{23}$.

10. Find the number of zeroes at the end of $(2016!)^{2016}$. Your answer should be an integer, not its prime factorization.

Solution: By Legendre's Formula, the number of zeroes at the end of 2016! is $\lfloor \frac{2016}{5} \rfloor + \lfloor \frac{2016}{25} \rfloor + \lfloor \frac{2016}{125} \rfloor + \lfloor \frac{2016}{625} \rfloor = 403 + 80 + 16 + 3 = 502$. It follows that our answer is $2016 \cdot 502 = \boxed{1,012,032}$

11. What is the sum of all positive integers less than 30 divisible by $2, 3,$ or 5?

Solution: It is much simpler to simply subtract the sum of the integers that are not divisible by $2, 3,$ or 5 from the sum of the first 29 positive integers. This gives us $\frac{29 \cdot 30}{2} - 1 - 7 - 11 - 13 - 17 - 19 - 23 - 29 = \boxed{315}$.

12. Let $g_0 = 1$, $g_1 = 2$, $g_2 = 3$, and $g_n = g_{n-1} + 2g_{n-2} + 3g_{n-3}$. For how many $0 \leq i \leq 100$ is it that g_i is divisible by 5?

Solution: Notice that the first few values of g leave remainders of $1, 2, 3, 0, 2, 1, 0, 3, 1, 2, 3, \cdots$. It follows that g repeats in cycles of length 8, and each of these cycles has 2 0's or multiples of 5 in them. It follows that in the first 96 terms, there are $12 \cdot 2 = 24$ multiples of 5. For $96 \leq i \leq 100$, there will be 1 multiple of 5 for a total of $\boxed{25}$ multiples of 5.

13. Define r_n to be the number of integer solutions to $x^2+y^2 = n$. Determine $\lim_{n\to\infty} \dfrac{r_1 + r_2 + ... + r_n}{n}$.

Solution: Notice that $r_1 + r_2 + \cdots + r_n$ is approximately the number of lattice points within a circle of radius $\sqrt{n}$ of the origin. This circle can be inscribed in a square of side length $2\sqrt{n}$ centered about the origin, and this square has approximately $4n$ lattice points inside it. The ratio of the area of the circle to the area of the square is $\frac{\pi}{4}$, so the number of lattice points inside the circle is approximately $\frac{\pi}{4} \cdot 4n = n\pi$. As n approaches ∞, these approximations will become exact with negligible errors, and therefore our fraction will approach $\frac{n\pi}{n} = \boxed{\pi}$.

14. How many integers less than 400 have exactly 3 factors that are perfect squares?

Solution: A number will have exactly 3 factors that are perfect squares if and only if its prime factors which divide the number more than once are in powers of the form p^4 or p^5. Therefore, we will do casework on what the value of p is.

Case 1: $p = 2$

In this case, we know our number n is divisible by 16 but not 64 and it is not divisible by the square of any other prime. There are 24 natural numbers less than 400 which are divisible by 16. 6 of these are divisible by 64, and 1 of these is divisible by 9. It follows that there are $24 - 6 - 1 = 17$ numbers in this case.

Case 2: $p = 3$

In this case, we know our number n is divisible by 81 but not by the square of any other prime. There are 4 multiples of 81 less than 400, but one of these is divisible by 4, so we have $4 - 1 = 3$ numbers in this case.

When $p > 3$, $p^4 > 400$, so these are our only two cases. Adding up our cases, we get an answer of $17 + 3 = \boxed{20}$.

15. For how many numbers n does 2017 divided by n have a remainder of either 1 or 2?

Solution: If $2017 \equiv 1 \pmod{n}$, then it follows that n is a factor of 2016 which is greater than 1. If $2017 \equiv 2 \pmod{n}$, then it follows that n is a factor of 2015 which is greater than 2. $2015 = 5^1 \cdot 13^1 \cdot 31^1$, so it follows that 2015 has 7 factors which are greater than 2. $2016 = 2^5 \cdot 3^2 \cdot 7^1$, so it follows that 2016 has 35 factors which are greater than 2. Therefore our answer is $7 + 35 = \boxed{42}$.

3 Sources

1. 2015 Berkeley Math Tournament Spring Individual Problem 6
2. 2015 Berkeley Math Tournament Spring Individual Problem 9
3. 2015 Berkeley Math Tournament Spring Discrete Problem 2
4. 2015 Berkeley Math Tournament Spring Discrete Problem 4
5. 2015 Berkeley Math Tournament Spring Discrete Problem 9
6. 2015 Berkeley Math Tournament Spring Team Problem 5
7. 2015 Berkeley Math Tournament Spring Team Problem 13
8. 2016 Berkeley Math Tournament Fall Individual Problem 17
9. 2016 Berkeley Math Tournament Fall Team Problem 14
10. 2016 Berkeley Math Tournament Fall Team Problem 16
11. 2016 Berkeley Math Tournament Spring Individual Problem 1
12. 2016 Berkeley Math Tournament Spring Individual Problem 6
13. 2016 Berkeley Math Tournament Spring Individual Problem 18
14. 2016 Berkeley Math Tournament Spring Team Problem 6
15. 2017 Berkeley Math Tournament Spring Individual Problem 6