Number Theory Handout #5

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1 Problems

1. An integer-valued function f satisfies $f(2) = 4$ and $f(mn) = f(m)f(n)$ for all integers m and n. If f is an increasing function, determine $f(2015)$.

2. The number 2^{29} has a 9-digit decimal representation that contains all but one of the 10 (decimal) digits. Determine which digit is missing.

3. Determine the largest integer n such that 2^n divides the decimal representation given by some permutation of the digits $2, 0, 1$, and 5. (For example, 2^1 divides 2150. It may start with 0.)

4. Determine the greatest integer N such that N is a divisor of $n^{13} - n$ for all integers n.

5. There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

6. Determine the smallest positive integer containing only 0 and 1 as digits that is divisible by each integer 1 through 9.

7. There exist right triangles with integer side lengths such that the legs differ by 1. For example, $3-4-5$ and $20-21-29$ are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

8. When 20^{16} is divided by 16^{20} and expressed in decimal form, what is the number of digits to the right of the decimal point? Trailing zeroes should not be included.

9. Let $2016 = a_1 \times a_2 \times \cdots \times a_n$ for some positive integers $a_1, a_2, ..., a_n$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_n$.

10. Find the number of zeroes at the end of $(2016!)^{2016}$. Your answer should be an integer, not its prime factorization.

11. What is the sum of all positive integers less than 30 divisible by $2, 3$, or 5?

12. Let $g_0 = 1$, $g_1 = 2$, $g_2 = 3$, and $g_n = g_{n-1} + 2g_{n-2} + 3g_{n-3}$. For how many $0 \leq i \leq 100$ is it that g_i is divisible by 5?

13. Define r_n to be the number of integer solutions to $x^2+y^2 = n$. Determine $\lim_{n\to\inf} \dfrac{r_1 + r_2 + ... + r_n}{n}$.

14. How many integers less than 400 have exactly 3 factors that are perfect squares?

15. For how many numbers n does 2017 divided by n have a remainder of either 1 or 2?

2 Sources

1. 2015 Berkeley Math Tournament Spring Individual Problem 6
2. 2015 Berkeley Math Tournament Spring Individual Problem 9
3. 2015 Berkeley Math Tournament Spring Discrete Problem 2
4. 2015 Berkeley Math Tournament Spring Discrete Problem 4
5. 2015 Berkeley Math Tournament Spring Discrete Problem 9
6. 2015 Berkeley Math Tournament Spring Team Problem 5
7. 2015 Berkeley Math Tournament Spring Team Problem 13
8. 2016 Berkeley Math Tournament Fall Individual Problem 17
9. 2016 Berkeley Math Tournament Fall Team Problem 14
10. 2016 Berkeley Math Tournament Fall Team Problem 16
11. 2016 Berkeley Math Tournament Spring Individual Problem 1
12. 2016 Berkeley Math Tournament Spring Individual Problem 6
13. 2016 Berkeley Math Tournament Spring Individual Problem 18
14. 2016 Berkeley Math Tournament Spring Team Problem 6
15. 2017 Berkeley Math Tournament Spring Individual Problem 6