Number Theory Handout #8

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1 Problems

1. When the integer $(\sqrt{3}+5)^{103}-(\sqrt{3}-5)^{103}$ is divided by 9, what is the remainder?

2. The value of 21! is $51,090,942,171,abc,440,000$ where a, b, and c are digits. What is the value of $100a+10b+c$?

3. Find the smallest two-digit positive integer that is a divisor of 201020112012.

4. When Meena turned 16 years old, her parents gave her a cake with n candles, where n has exactly 16 different positive integer divisors. What is the smallest possible value of n?

5. The number $104,060,465$ is divisible by a five-digit prime number. What is that prime number?

6. Let N be the number of ordered pairs of integers (x, y) such that

$$4x^2+9y^2 \leq 1000000000.$$

Let a be the first digit of N (from the left) and let b be the second digit of N. What is the value of $10a+b$?

7. The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3)=89$, what is the value of $P(10)$?

8. What is the least positive integer n such that $n!$ is a multiple of 2012^{2012}?

9. For how many ordered pairs of positive integers (x, y) is the least common multiple of x and y equal to $1,003,003,001$?

10. When the binomial coefficient $\binom{125}{64}$ is written out in base 10, how many zeros are at the rightmost end?

11. If n is a positive integer, let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n-2^n}.$$

Express your answer as a fraction in simplest form.

12. Say that an integer A is *yummy* if there exist several consecutive integers (including A) that add up to 2014. What is the smallest yummy integer?

13. Say that an integer $n \geq 2$ is *delicious* if there exist n positive integers adding up to 2014 that have distinct remainders when divided by n. What is the smallest delicious integer?

14. There are N students in a class. Each possible nonempty group of students selected a positive integer. All of these integers are distinct and add up to 2014. Compute the greatest possible value of N.

15. For how many integers k such that $0 \leq k \leq 2014$ is it true that the binomial coefficient $\binom{2014}{k}$ is a multiple of 4?

2 Sources

1. Math Prize For Girls 2009 Problem 9
2. Math Prize For Girls 2009 Problem 18
3. Math Prize For Girls 2010 Problem 5
4. Math Prize For Girls 2010 Problem 8
5. Math Prize For Girls 2011 Problem 13
6. Math Prize For Girls 2011 Problem 16
7. Math Prize For Girls 2011 Problem 18
8. Math Prize For Girls 2012 Problem 3
9. Math Prize For Girls 2012 Problem 6
10. Math Prize For Girls 2013 Problem 2
11. Math Prize For Girls 2013 Problem 19
12. Math Prize For Girls 2014 Problem 4
13. Math Prize For Girls 2014 Problem 5
14. Math Prize For Girls 2014 Problem 6
15. Math Prize For Girls 2014 Problem 18