Geometry Handout 2 Answers and Solutions

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October 17, 2017

1 Answers

1. $\frac{50}{3}$

2. $\frac{625\pi}{51}$

3. $2\sqrt{5}$

4. $\frac{39\pi}{4}$

5. $\sqrt{2}$

6. $\sqrt{13}$

7. 61

8. 144

9. $2\sqrt{5}-4$

10. 74

11. 45°

12. 17 and 35

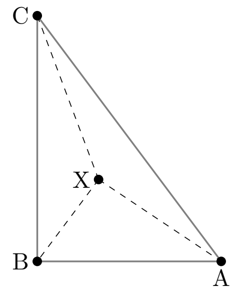
13. $\frac{2}{5}$

14. $2\sqrt{3}$

15. 12

2 Solutions

1. Let $\triangle ABC$ be a triangle such that $\overline{AB}=3$, $\overline{BC}=4$, and $\overline{AC}=5$. Let X be a point in the triangle. Compute the minimal possible value of $\overline{AX}^2+\overline{BX}^2+\overline{CX}^2$.

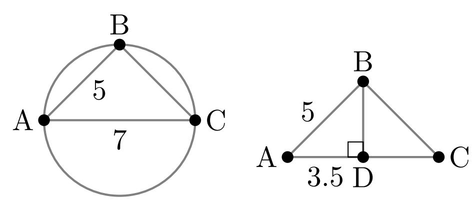


Solution: We will proceed with coordinates. Let the coordinates of B be $(0,0)$, the coordinates of A be $(3,0)$, and the coordinates of C be $(0,4)$. Let the coordinates of X be (a,b). Notice that by the Distance Formula, $\overline{AX}^2 = (3-a)^2+b^2$. Continuing with this formula, we find that the expression we wish to minimize is

$$(3-a)^2+a^2+a^2+(4-b)^2+b^2+b^2 = (3a^2-6a)+(3b^2-8b)+25 = 3(a-1)^2+3(b-\frac{4}{3})^2+\frac{50}{3}$$

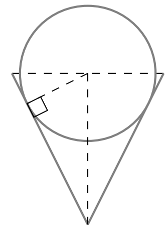
By the Trivial Inequality, the minimum value of this expression is $\boxed{\frac{50}{3}}$, when the coordinates of X are $(1, \frac{4}{3})$.

2. In a circle, chord $\overline{AB}$ has length 5 and chord $\overline{AC}$ has length 7. Arc $\overset{\frown}{AC}$ is twice the length of arc $\overset{\frown}{AB}$ and both arcs have angles less than 180 degrees. Compute the area of the circle.



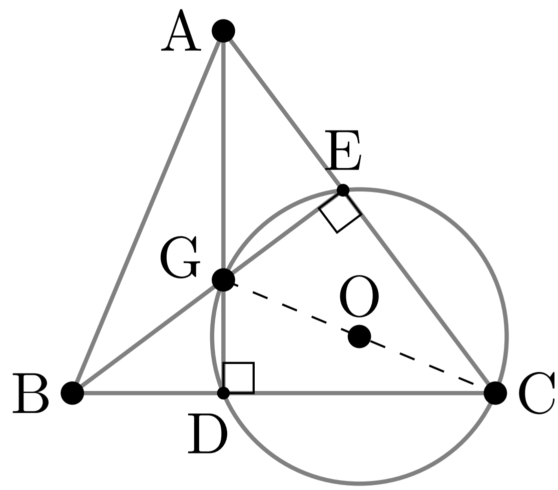
Solution: Notice that if arc $\overset{\frown}{AC}$ is twice the length of arc $\overset{\frown}{AB}$, the length of chord $\overline{AB}$ is the same as the length of chord $\overline{BC}$. Therefore, the circle is the circumcircle of a triangle with side lengths 5, 5, and 7. Let D be the foot of the perpendicular from B to $\overline{AC}$. Because $\overline{AB} \cong \overline{BC}$, we have that $\overline{AD} \cong \overline{DC} = 3.5$. By the Pythagorean theorem, it follows that $\overline{BD} = \frac{\sqrt{51}}{2}$, and that the area of $\triangle ABC$ is $\frac{7\sqrt{51}}{4}$. Now, remembering that $\frac{abc}{4R} = A$, where A is the area of the triangle and R is the circumradius of the triangle, we get that $R = \frac{5\cdot5\cdot7}{7\sqrt{51}} = \frac{25}{\sqrt{51}}$. It follows that the area of the circle is $R^2\pi = \boxed{\frac{625\pi}{51}}$ as desired.

3. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencers ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.



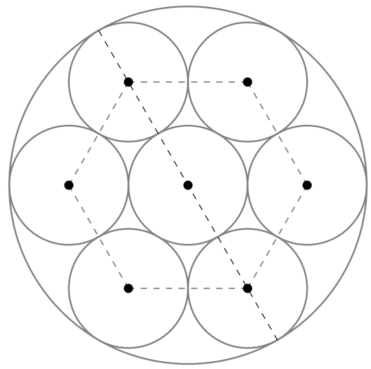
Solution: Taking a 2D cross-section of the problem as shown above, we get that we wish to inscribe a semicircle in an isosceles triangle with a base of length 10 and a height of length 10. It follows that the maximum radius is the height to the hypotenuse of a right triangle with legs of length 5 and 10. It follows that the radius is $\dfrac{5 \cdot 10}{\sqrt{5^2 + 10^2}} = \frac{10}{\sqrt{5}} = \boxed{2\sqrt{5}}$ as desired.

4. Let $\triangle ABC$ be a triangle with $\overline{AB} = 13$, $\overline{BC} = 14$, and $\overline{AC} = 15$. Let D and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of triangle $\triangle CDE$.



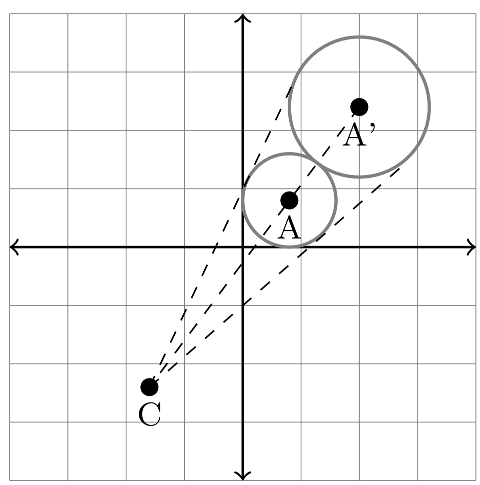
Solution: Let the intersection of $\overline{BE}$ and $\overline{AD}$ be G as shown above. Notice that because $\angle GDC + \angle BEC = 90^\circ + 90^\circ = 180^\circ$, $DGEC$ is a cyclic quadrilateral, and therefore the circumcircle of $\triangle CDE$ passes through G. Notice that because $\triangle GDC$ is a right triangle, the circumcircle of $\triangle GDC$ is centered at the midpoint of GC. By Heron's Formula, we have that the area of $\triangle ABC$ is $\sqrt{21 \cdot (21-13) \cdot (21-14) \cdot (21-15)} = 84$. It follows that the length of AD is $\frac{2 \cdot 84}{14} = 12$. From here, we can get that $\overline{BD} = \sqrt{13^2 - 12^2} = 5$ and $\overline{DC} = 14 - 5 = 9$ by the Pythagorean Theorem. Similarly, we can get that $\overline{BE} = \frac{2 \cdot 84}{15} = \frac{56}{5}$, and it follows that $\overline{EC} = \sqrt{14^2 - (\frac{56}{5})^2} = \frac{42}{5}$ and $\overline{AE} = 15 - \frac{42}{5} = \frac{33}{5}$. Proceeding by Mass Points, if we place a mass of 9 at B, then we must place a mass of $5 \cdot \frac{9}{9}$ at C, and a mass of $5 \cdot \frac{42}{33} = \frac{70}{11}$ at A. Finally, we must place a mass of $5 + 9 = 14$ at D. It follows that the ratio $\frac{\overline{AG}}{\overline{GD}} = \frac{14}{\frac{70}{11}} = \frac{11}{5}$. It follows that $\overline{GD} = \frac{5}{16} \cdot 12 = \frac{15}{4}$. Therefore, the length of $\overline{GC}$ is $\sqrt{(\frac{15}{4})^2 + 9^2} = \frac{39}{4}$. It follows that the circumference of the circumcircle of triangle $\triangle CDE$ is $\boxed{\dfrac{39\pi}{4}}$ as desired.

5. Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighboring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookie of the same thickness. What is the radius in inches of the scrap cookie?



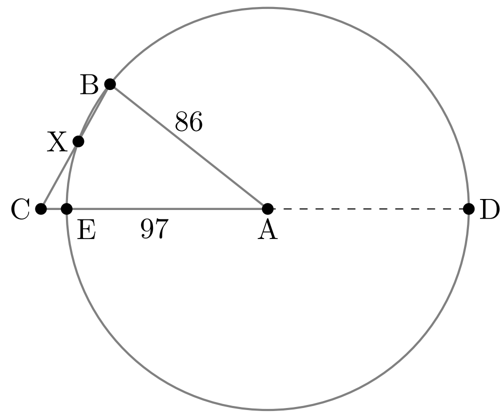
Solution: Notice that if we connect the centers of all the outside cookies, we get a regular hexagon with a side length of 2. Remembering that the inner diagonal of a hexagon is double its side length, we can find that the diameter of the circle of cookie dough is $2 \cdot 2 + 2 \cdot 1 = 6$. Therefore, the area of the cookie dough is $\pi(3)^2 = 9\pi$. Subtracting the areas of the original cookies, we get a new area of $9\pi - 7 \cdot \pi = 2\pi$. This will create a new cookie with a radius of $\sqrt{\frac{2\pi}{\pi}} = \boxed{\sqrt{2}}$ as desired.

6. A dilation of the plane - that is, a size transformation with a positive scale factor - sends the circle of radius 2 centered at A $(2,2)$ to the circle of radius 3 centered at A' $(5,6)$. What distance does the origin O $(0,0)$, move under this transformation?



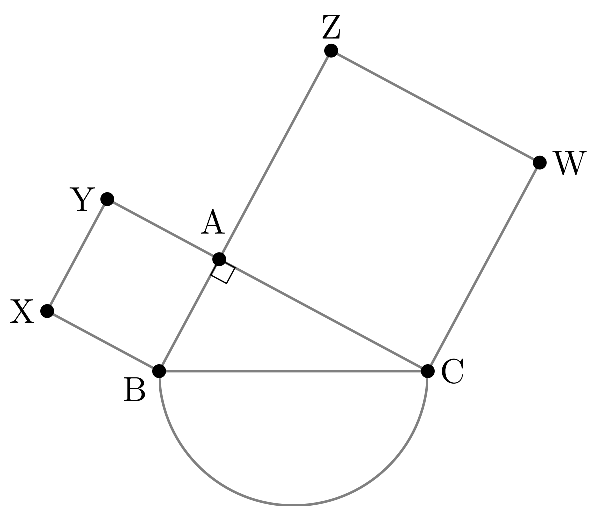
Solution: Because a circle of radius 2 was dilated to a circle of radius 3, we know that the scale factor is $\frac{3}{2} = 1.5$. If we let the center of the dilation be C as shown, then we know that $\frac{\overline{CA'}}{\overline{CA}} = \frac{3}{2}$. Because these three points are collinear, we can add double the displacement from A' to A to the coordinates of point A to get the coordinates of point C. The displacement from A' to A is $5-2=3$ in the x-direction and $6-2=4$ in the y-direction. It follows that the coordinates of point C are $(2-2\cdot 3, 2-2\cdot 4) = (-4,-6)$. The distance $\overline{OC}$ is $\sqrt{4^2+6^2} = 2\sqrt{13}$ by the distance formula. It follows that the displacement of O under this transformation is $\frac{2\sqrt{13}}{2} = \boxed{\sqrt{13}}$ as desired.

7. In triangle $\triangle ABC$, AB $= 86$, and $\overline{AC} = 97$. A circle with center A and radius $\overline{AB}$ intersects $\overline{BC}$ at points B and X. Moreover, $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is $\overline{BC}$?



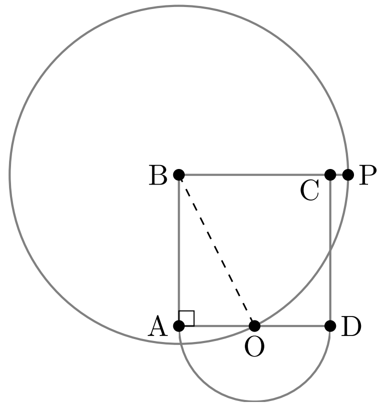
Solution: Let line CA intersect the circle at points E and D as shown. By Power of a Point on point C, we have that $\overline{CE}\cdot\overline{CD} = \overline{CX}\cdot\overline{CB}$. It follows that $\overline{CX}\cdot\overline{CB} = (97-86)\cdot(97+86) = 2013$. Factorizing 2013, we can find that the only pairs which work for $(\overline{CX}, \overline{BC})$ are $(1, 2013), (3, 671), (11, 183)$, and $(33, 61)$. Because $11 < \overline{BC}$ and $\overline{BC} < 183$ by the Triangle Inequality, we must have that $(\overline{CX}, \overline{BC})$ is $(33, 61)$. Therefore, $\overline{BC} = \boxed{61}$ as desired.

8. Right triangle $\triangle ABC$ has squares $ABXY$ and $ACWZ$ drawn externally to its legs and a semicircle drawn externally to its hypotenuse $\overline{BC}$. If the area of the semicircle is 18π and the area of triangle $\triangle ABC$ is 30, what is the sum of the areas of squares $ABXY$ and $ACWZ$?



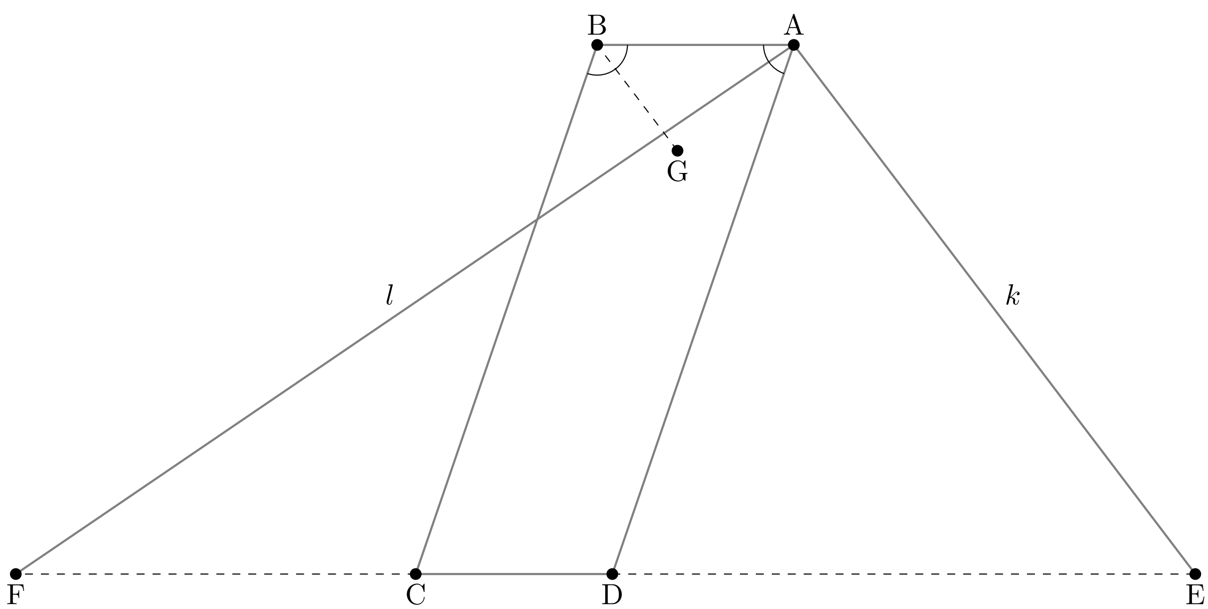
Solution: If the area of the semicircle is 18π, then by $A = \pi r^2$, we must have that $R^2 = 2 \cdot 18$, where R is the radius of the semicircle. It follows that $\overline{BC}$ is $2 \cdot 6 = 12$. By the Pythagorean Theorem, we know that $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$. It follows that the sum of the areas of squares $ABXY$ and $ACWZ$ is $12^2 = \boxed{144}$ as desired.

9. Let O be the center of a semicircle with diameter $\overline{AD}$ and area 2π. Given square $ABCD$ drawn externally to the semicircle, construct a new circle with center B and radius $\overline{BO}$. If we extend $\vec{BC}$, this new circle intersects $\vec{BC}$ at P. What is the length of $\overline{CP}$?



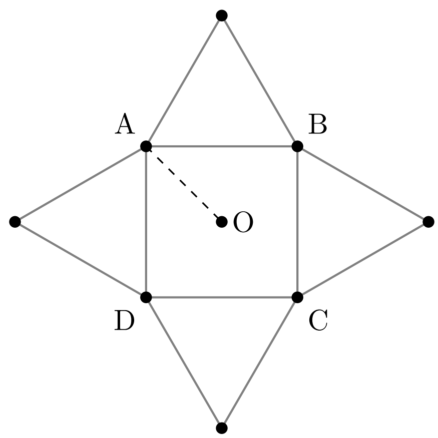
Solution: Consider the semicircle. If its area is 2π, by $A = \pi r^2$, we have that $R^2 = 2 \cdot 2 = 4$, where R is the radius of the semicircle. It follows that $\overline{AD} = 4$ and $\overline{AO} = 2$. Noticing that $\angle BAO = 90°$, we can find that by the Pythagorean Theorem, $\overline{BO} = 2\sqrt{5}$. It follows that $\overline{CP} = \overline{BP} - \overline{BC} = \boxed{2\sqrt{5} - 4}$ as desired.

10. Consider a parallelogram $ABCD$. Let k be the line passing through A and parallel to the bisector of angle $\angle ABC$, and let l be the bisector of angle $\angle BAD$. Let k intersect line $\overline{CD}$ at E and l intersect line $\overline{CD}$ at F. If $\overline{AB} = 13$ and $\overline{BC} = 37$, find the length $\overline{EF}$.



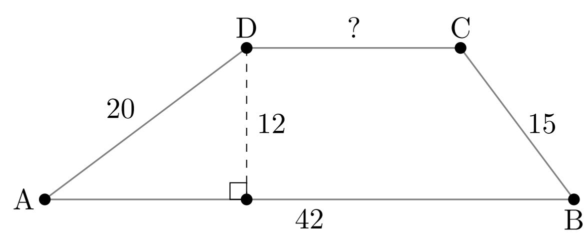
Solution: Notice that by the parallel lines, we have $\angle DAF \cong \angle BAF \cong \angle AFD$. It follows that $\triangle AFD$ is isosceles and $\overline{FD} \cong \overline{AD} = 37$. Similarly, we have $\angle AED \cong \angle ABE \cong \angle CBE \cong DAE$. It follows that $\overline{DE} \cong \overline{AD} = 37$. Therefore, $\overline{EF} = \overline{FD} + \overline{DE} = \boxed{74}$ as desired.

11. Square $ABCD$ has equilateral triangles drawn external to each side, as pictured. If each triangle is folded upwards to meet at a point E, then a square pyramid can be made. If the center of square $ABCD$ is O, what is the measure of angle $\angle OEA$?



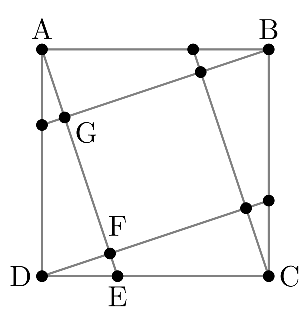
Solution: Let O be at the center of $ABCD$ as shown. By symmetry, we must have $\angle EOA$ is a right angle. It follows that $\sin \angle OEA = \frac{\overline{AO}}{\overline{EA}} = \frac{\frac{s\sqrt{2}}{2}}{s} = \frac{\sqrt{2}}{2}$. It follows that $\angle OEA = \boxed{45^\circ}$ as desired.

12. A trapezoid with height 12 has legs of length 20 and 15 and a larger base of length 42. What are the possible lengths of the other base?



Solution: Label the points A, B, C, and D as shown. Let $\overline{AB}$ lie on the x-axis. It follows that the displacement in the x-direction from A to D is $\sqrt{20^2 - 12^2} = 16$ and the displacement in the x-direction from B to C is $\sqrt{15^2 - 12^2} = 9$. It follows that $\overline{DC} = 42 \pm 16 \pm 9$. Of all of the possible values of this expression, the only values which are less than 42 are $42 - 16 - 9 = 17$ and $42 - 16 + 9 = 35$. Therefore, our answer is $\boxed{17 \text{ and } 35}$ as desired.

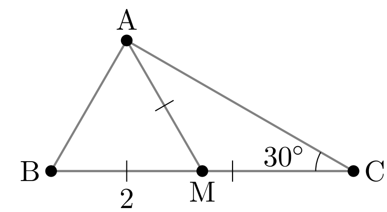
13. As pictured, lines are drawn from the vertices of a unit square to an opposite trisection point. If each triangle has legs with ratio $3:1$, what is the area of the slanted square formed in the middle of $ABCD$?



Solution: Label the points E, F and G as shown above. By the Pythagorean Theorem, we have $\overline{AE} = \sqrt{(\frac{1}{3})^2 + 1} = \frac{\sqrt{10}}{3}$. Notice that because $\angle FED \cong \angle DEA$ and $\angle EFD \cong EDA$, we have

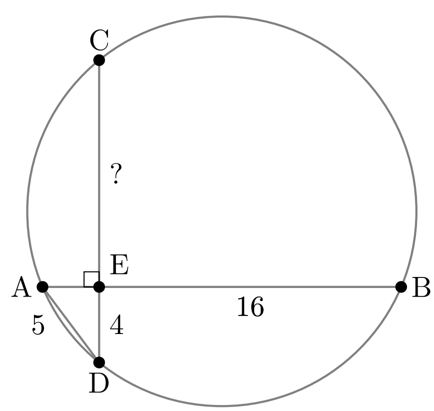
$\triangle ADE$ is similar to $\triangle DFE$. It follows that $\frac{\overline{EF}}{\overline{DE}} = \frac{\overline{DE}}{\overline{AE}}$. Solving, we can determine $\overline{EF} = \frac{\sqrt{10}}{30}$. By the Pythagorean Theorem, we can determine $\overline{DF} = \sqrt{(\frac{1}{3})^2 - (\frac{\sqrt{10}}{30})^2} = \frac{\sqrt{10}}{10}$. By symmetry, we have $\overline{AG} \cong \overline{DF} = \frac{\sqrt{10}}{10}$. It follows that $\overline{FG} = \frac{\sqrt{10}}{3} - \frac{\sqrt{10}}{10} - \frac{\sqrt{10}}{30} = \frac{\sqrt{10}}{5}$. It follows that the area of the square is $(\frac{\sqrt{10}}{5})^2 = \boxed{\frac{2}{5}}$ as desired.

14. Let $\triangle ABC$ be a triangle. Let M be the midpoint of $\overline{BC}$. Suppose $\overline{MA} = \overline{MB} = \overline{MC} = 2$ and angle $\angle ACB = 30°$. Find the area of the triangle.



Solution: Noticing that $\triangle AMC$ is isosceles, we have $\angle AMC = 180° - 2 \cdot 30° = 120°$. It follows that $\angle AMB = 60°$, and $\triangle AMB$ is equilateral. Therefore, $\angle BAM = 60°$ and $\angle MAC = 30°$, so $\angle BAC = 90°$ and $\triangle ABC$ is a $30 - 60 - 90$ triangle. It follows that $\overline{AB} = 2$ and $\overline{AC} = 2\sqrt{3}$. Therefore, our answer is $\frac{2 \cdot 2\sqrt{3}}{2} = \boxed{2\sqrt{3}}$ as desired.

15. Chords $\overline{AB}$ and $\overline{CD}$ of a given circle are perpendicular to each other and intersect at a right angle E. Given that $\overline{BE} = 16$, $\overline{DE} = 4$, and $\overline{AD} = 5$, find $\overline{CE}$.



Solution: Notice that by the Pythagorean Theorem, $\overline{AE} = \sqrt{5^2 - 4^2} = 3$. By Power of a Point on point E, we have that $\overline{AE} \cdot \overline{EB} = \overline{ED} \cdot \overline{CE}$. It follows that $\overline{CE} = \frac{3 \cdot 16}{4} = \boxed{12}$ as desired.

3 Sources

1. 2014 Stanford Math Tournament General 25
2. 2014 Stanford Math Tournament Geometry 2
3. 2014 Stanford Math Tournament Geometry 3
4. 2014 Stanford Math Tournament Geometry 7

5. 2016 AMC 10A Problem 15
6. 2016 AMC 10B Problem 20
7. 2013 AMC 10A Problem 23
8. Berkeley Math Tournament Fall 2012 Individual Problem 4
9. Berkeley Math Tournament Fall 2012 Individual Problem 9
10. Berkeley Math Tournament Fall 2012 Individual Problem 19
11. Berkeley Math Tournament Fall 2012 Team Problem 3
12. Berkeley Math Tournament Fall 2012 Team Problem 5
13. Berkeley Math Tournament Fall 2012 Team Problem 8
14. Berkeley Math Tournament Fall 2013 Individual Problem 17
15. Berkeley Math Tournament Fall 2013 Speed Problem 75