Geometry Handout #6 Answers and Solutions

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1 Answers

1. 769

2. $67°$

3. $130°$

4. $\frac{1}{3}$

5. $\frac{35}{2}$

6. $\frac{13}{2}$

7. 2

8. $51 - 36\sqrt{2}$

9. 31

10. 31

11. $3 - 2\sqrt{2}$

12. 74

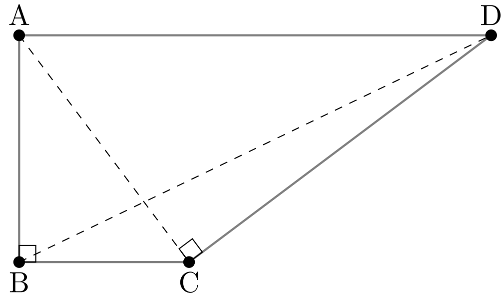
13. $\frac{12}{5}$

14. $\frac{4+\sqrt{2}+\sqrt{6}}{2}$

15. $\frac{6+3\sqrt{3}}{2}$

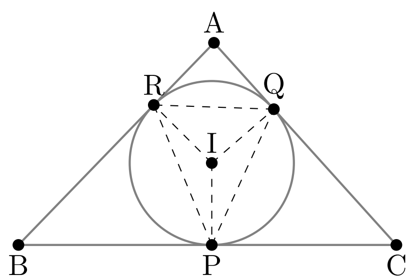
2 Solutions

1. A quadrilateral $ABCD$ has a right angle at $\angle ABC$ and satisfies $\overline{AB} = 12$, $\overline{BC} = 9$, $\overline{CD} = 20$, and $\overline{DA} = 25$. Determine $\overline{BD}^2$.



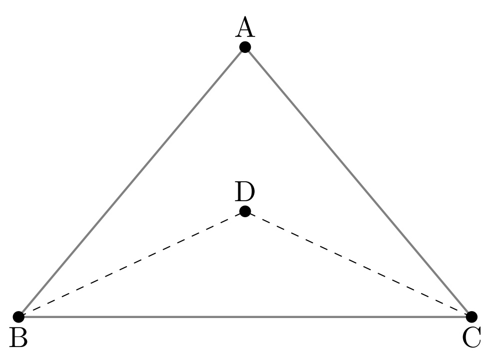
Solution: Using the Pythagorean Theorem on $\triangle ABC$, we can easily find that $\overline{AC} = 15$. Because $\overline{CD}^2 + \overline{AC}^2 = 20^2 + 15^2 = 25^2 = \overline{AD}^2$, it follows by the Pythagorean Theorem that $\angle ACD = 90°$. It follows that $\sin BAC = \frac{3}{5}$ and that $\cos DAC = \frac{3}{5}$. It follows that $\angle DAB = \angle BAC + \angle DAC = 90°$. Therefore, by the Pythagorean Theorem we know that $\overline{BD}^2 = \overline{AD}^2 + \overline{AB}^2 = 25^2 + 12^2 = \boxed{769}$.

2. In $\triangle ABC$, $m\angle B = 46°$ and $m\angle C = 48°$. A circle is inscribed in $\triangle ABC$ and the points of tangency are connected to form $\triangle PQR$. What is the measure of the largest angle in $\triangle PQR$?



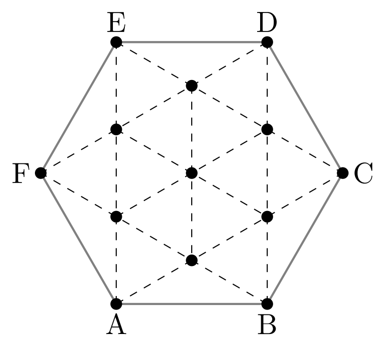
Solution: Let I be the incenter of $\triangle ABC$. By definition, we know that $\angle PIQ = 2\angle QRP, \angle QIR = 2\angle QPR$, and $\angle PIR = 2\angle PQR$. However, we also know that $\angle IPB \cong \angle IPC \cong \angle IQC \cong \angle IQA \cong \angle IRA \cong \angle IRB = 90°$. It follows that $\angle RIQ = 180° - \angle BAC = 46 + 48 = 94°$, $\angle RIP = 180° - \angle ABC = 180 - 46 = 134°$, and $\angle QIP = 180° - \angle ACB = 180 - 48 = 132°$. It follows that the largest of these three angles is $\angle RIP = 134°$, and therefore our answer is $\frac{\angle RIP}{2} = \frac{134}{2} = \boxed{67°}$.

3. Let $\triangle ABC$ be a triangle. The angle bisectors of $\angle ABC$ and $\angle ACB$ intersect at D. If $\angle BAC = 80°$, what are all possible values for $\angle BDC$?



Solution: Notice that $\angle BDC = 180° - \angle DBC - \angle DCB = 180° - \frac{\angle ABC + \angle ACB}{2} = 180° - \frac{180° - \angle BAC}{2} = 180° - \frac{180° - 80°}{2} = 180° - 50° = 130°$. Therefore, the only possible value of $\angle BDC$ is $\boxed{130°}$.

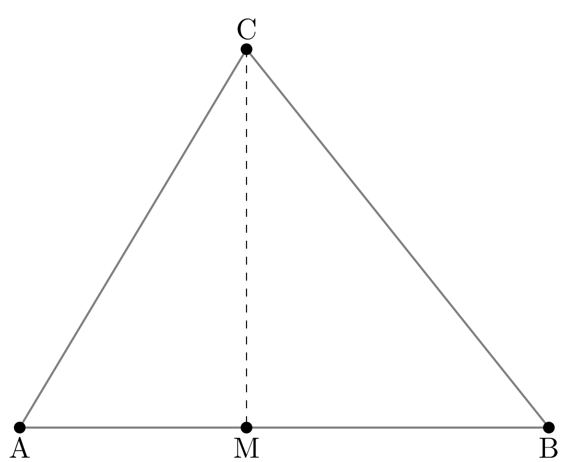
4. $ABCDEF$ is a regular hexagon. Let R be the overlap between $\triangle ACE$ and $\triangle BDF$. What is the area of R divided by the area of $ABCDEF$?



Solution: Notice that by symmetry, R is a regular hexagon. By connecting the diagonals of R, we can partition $\triangle BDF$ into 9 congruent equilateral triangles. It follows that the area of R is $\frac{6}{9} = \frac{2}{3}$ of the area of $\triangle BDF$. If we let the side length of $ABCDEF$ be s, then it follows that because $\triangle ABF$ is an isosceles triangle with congruent sides of length s and $\angle BAF$ is $120°$, we must have that $\overline{BF} = s\sqrt{3}$. It follows that the area of $\triangle BDF$ is $\frac{(s\sqrt{3})^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{4}$. We know that the area of $ABCDEF$ is $\frac{3s^2\sqrt{3}}{2}$, so our answer is

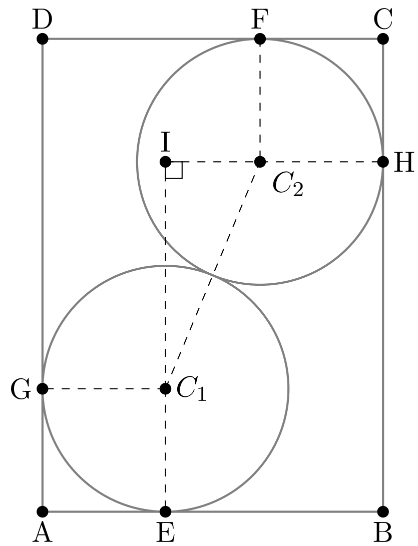
$$\frac{\frac{2}{3} \cdot \frac{3s^2\sqrt{3}}{4}}{\frac{3s^2\sqrt{3}}{2}} = \frac{\frac{2}{4}}{\frac{3}{2}} = \frac{4}{12} = \boxed{\frac{1}{3}}$$

5. Let M be on segment BC of $\triangle ABC$ so that $\overline{AM} = 3$, $\overline{BM} = 4$, and $\overline{CM} = 5$. Find the largest possible area of $\triangle ABC$.



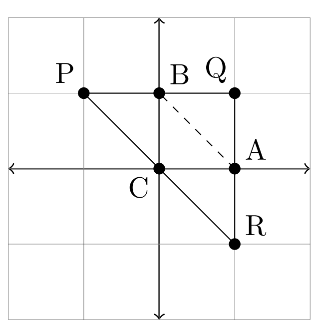
Solution: Consider the length of the altitude from C to $\overline{AB}$, and call this length l. We know that this is the minimum length from C to any point on $\overline{AB}$, so we know that $l \leq \overline{CM} = 5$. We also know that the area of $\triangle ABC$ is $\frac{l\cdot(\overline{AM}+\overline{BM})}{2} = \frac{7l}{2}$. Clearly this area will be maximized when $l = \overline{CM} = 5$, so our answer is $\frac{7\cdot 5}{2} = \boxed{\frac{35}{2}}$.

6. Let $ABCD$ be a rectangle. Circles C_1 and C_2 are externally tangent to each other. Furthermore, C_1 is tangent to $\overline{AB}$ and $\overline{AD}$, and C_2 is tangent to $\overline{CB}$ and $\overline{CD}$. If $\overline{AB} = 18$ and $\overline{BC} = 25$, then find the sum of the radii of the circles.



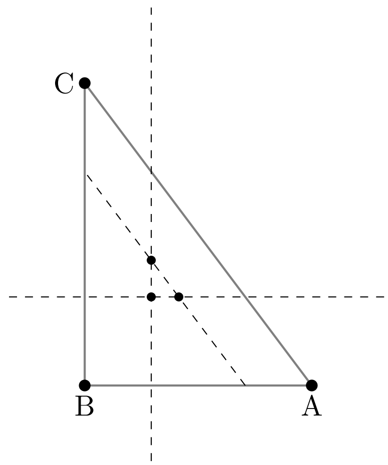
Solution: Label the diagram as shown above. Let the radius of each circle be r. Then we know that $25 = \overline{AD} = \overline{C_1E} + \overline{C_1I} + \overline{FC_2} = r + \overline{C_1I} + r$, or $\overline{C_1I} = 25 - 2r$. We know that $18 = \overline{CD} = \overline{GC_1} + \overline{IC_2} + \overline{C_2H} = r + \overline{IC_2} + r$, or $IC_2 = 18 - 2r$. Finally, we know that $\overline{C_1C_2} = 2r$. Using the Pythagorean Theorem on $\triangle IC_1C_2$, we can find that $(18-2r)^2 + (25-2r)^2 = (2r)^2$. Simplifying, we get $949 - 86r + 4r^2 = 0$. Solving, we get $r = \boxed{\frac{13}{2}}$.

7. Let $A = (1,0), B = (0,1)$, and $C = (0,0)$. There are three distinct points, P, Q, R, such that $\{A,B,C,P\}, \{A,B,C,Q\}, \{A,B,C,R\}$ are all parallelograms (vertices unordered). Find the area of $\triangle PQR$.



Solution: It is well known that the two diagonals of any parallelogram bisect eachother, or in other words, they intersect at the midpoint of each diagonal. Reflecting each of A, B, and C about the opposite segment of $\triangle ABC$ gives us $P = (-1, 1)$, $Q = (1, 1)$, and $R = (1, -1)$. It follows that the area of $\triangle PQR$ is $\dfrac{2 \cdot 2}{2} = \boxed{2}$.

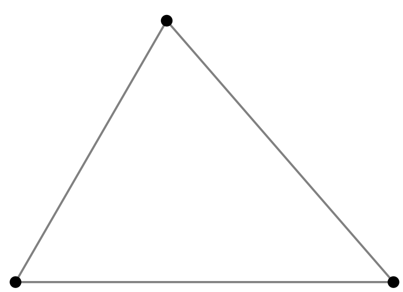
8. Triangle $\triangle ABC$ has side lengths $\overline{AB} = 3, \overline{BC} = 4$, and $\overline{CD} = 5$. Draw line l_A such that l_A is parallel to $\overline{BC}$ and splits the triangle into two polygons of equal area. Define lines l_B and l_C analogously. The intersection points of l_A, l_B, and l_C form a triangle. Determine its area.



Solution: We will proceed with coordinates. Let B be the origin, let A be $(3, 0)$, and let C be $(0, 4)$. We can easily see with similar triangles that the line l_A is represented by $x = 3 - \frac{3\sqrt{2}}{2}$ and the line l_C is represented by $y = 4 - 2\sqrt{2}$. Finally, we can easily see that l_B passes through the points $(0, 2\sqrt{2})$ and $(\frac{3\sqrt{2}}{2}, 0)$. It follows that l_B can be represented by $y = 2\sqrt{2} - \frac{4}{3}x$. It follows that the bottom left corner of our new triangle has coordinates $(3 - \frac{3\sqrt{2}}{2}, 4 - 2\sqrt{2})$, the bottom right corner of our new triangle has coordinates $(3\sqrt{2} - 3, 4 - 2\sqrt{2})$, and the top left corner of our new triangle has coordinates $(3 - \frac{3\sqrt{2}}{2}, 4\sqrt{2} - 4)$. It follows that the area of our new triangle is

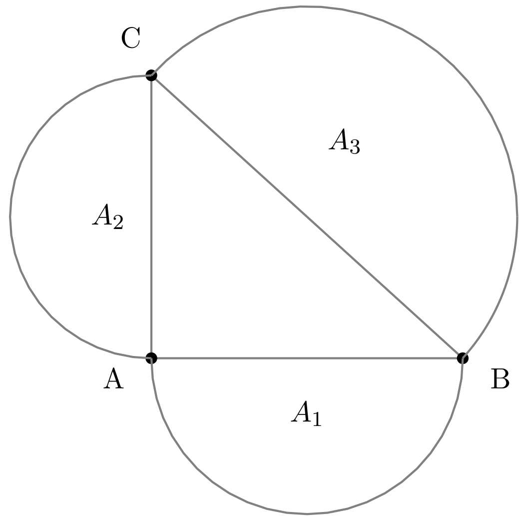
$$\frac{4\sqrt{2} - 4 - (4 - 2\sqrt{2})}{2} \cdot \frac{3\sqrt{2} - 3 - (3 - \frac{3\sqrt{2}}{2})}{1} = \boxed{51 - 36\sqrt{2}}$$

9. Suppose that two of the three sides of an acute triangle have lengths 20 and 16, respectively. How many possible integer values are there for the length of the third side?



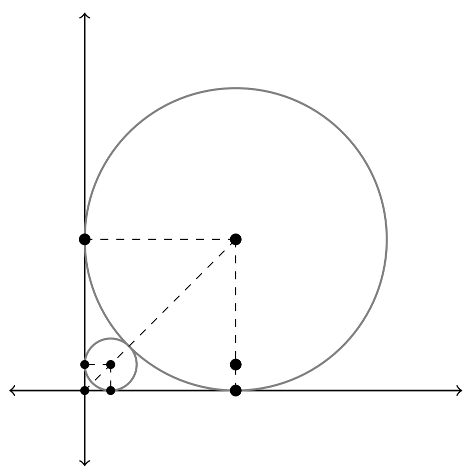
Solution: By the Triangle Inequality, if we let the third side have length s, then we know that $16 + 20 > s$ and $s + 16 > 20$. It follows that $5 \leq s \leq 35$. Therefore our answer is $35 - 5 + 1 = \boxed{31}$.

10. In the figure below, three semicircles are drawn outside the given right triangle. Given the areas $A_1 = 17$ and $A_2 = 14$, find the area A_3.



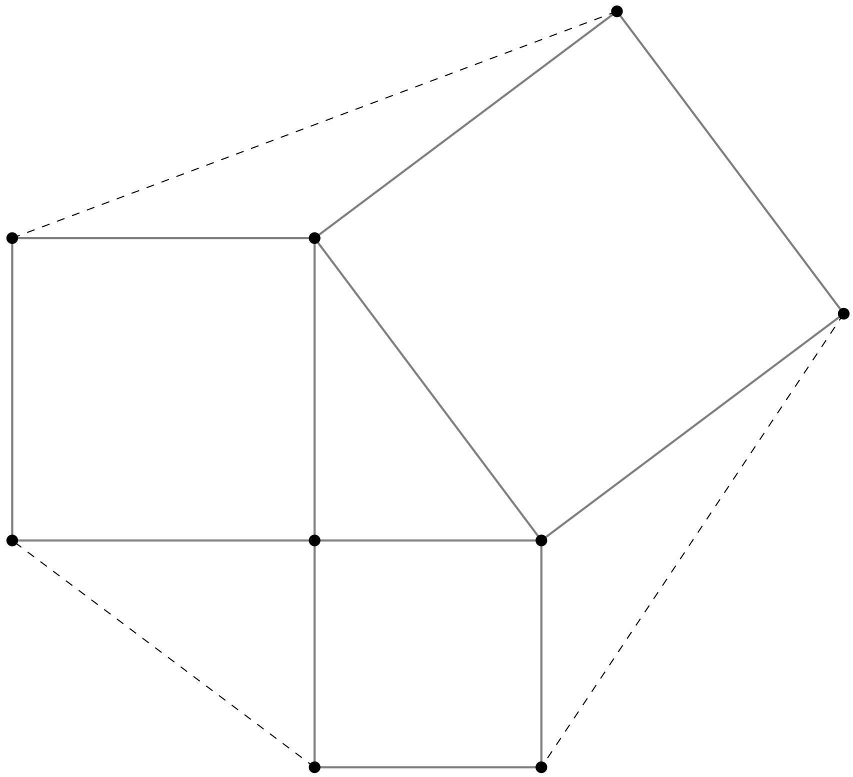
Solution: Label the vertices of the right triangle A, B, and C as shown above. Let $\overline{AB} = c$, let $\overline{AC} = b$, and let $\overline{BC} = a$. By the Pythagorean Theorem, we know that $b^2 + c^2 = a^2$. However, we also know that $\frac{c^2\pi}{8} = 17$ and $\frac{b^2\pi}{8} = 14$. It follows that $A_3 = \frac{a^2\pi}{8} = \frac{(b^2+c^2)\pi}{8} = 17 + 14 = \boxed{31}$.

11. Consider a circle of radius 1 drawn tangent to the positive x and y axes. Now consider another smaller circle tangent to that circle and also tangent to the positive x and y axes. Find the radius of the smaller circle.



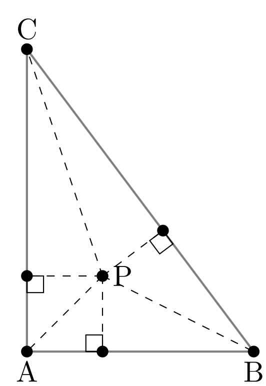
Solution: Let the radius of the smaller circle be r. Consider the square by connecting the center of the larger circle to each of its points of tangency. We know that the length of this diagonal is $d = \sqrt{2}$. However, we can also say that $d = 1 + r + r\sqrt{2}$ by considering the square formed by connecting the center of the smaller circle to its points of tangency. It follows that $r(1 + \sqrt{2}) = \sqrt{2} - 1$, and therefore our answer is $r = (\sqrt{2} - 1)^2 = \boxed{3 - 2\sqrt{2}}$.

12. Suppose you have a triangle with side lengths $3, 4,$ and 5. For each of the triangle's sides, draw a square on its outside. Connect the adjacent vertices in order, forming 3 new triangles (as in the diagram). What is the area of this convex region?



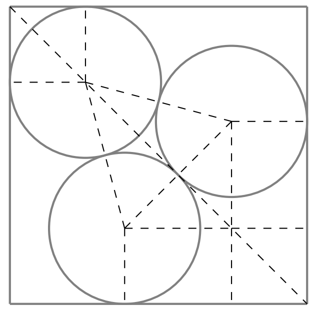
Solution: We can easily find that the sum of the areas of the three squares and the original right triangle is $\frac{3 \cdot 4}{2} + 3^2 + 4^2 + 5^2 = 56$. Using the facts that the area of a triangle is $\frac{ab \sin C}{2}$ and $\sin A = \sin(180° - A)$, we can find that the areas of the three outer triangles are $\frac{3 \cdot 4 \cdot \sin 90°}{2} = 6$, $\frac{3 \cdot 5 \cdot \frac{4}{5}}{2} = 6$, and $\frac{4 \cdot 5 \cdot \frac{3}{5}}{2} = 6$. Therefore our answer is $56 + 6 + 6 + 6 = \boxed{74}$.

13. Let $\triangle ABC$ have side lengths $3, 4,$ and 5. Let P be a point inside $\triangle ABC$. What is the minimum sum of the lengths of the altitudes from P to the side lengths of $\triangle ABC$?



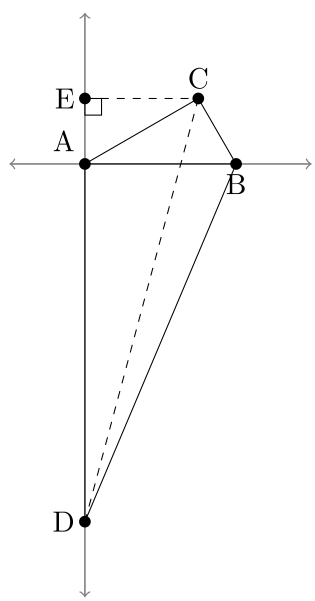
Solution: Notice that the area of $\triangle ABC$ is equivalent to the sum of the areas of triangles $\triangle APB$, $\triangle APC$, and $\triangle BPC$. If we let the altitudes from P to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ be $a, b,$ and c, respectively, then we know that $\frac{5a+4b+3c}{2} = 6$. It follows that $a + b + c$ will be minimized when $b = c = 0$ and $\frac{5a}{2} = 6$, or $a = \frac{12}{5}$. This can be achieved when P is at point A. Therefore, our answer is $0 + 0 + \frac{12}{5} = \boxed{\frac{12}{5}}$.

14. Three circles of radius 1 are inscribed in a square of side length s such that the circles do not overlap or coincide with each other. What is the minimum s where such a configuration is possible?



Solution: Clearly the configuration which will minimize the side of the square is the configuration above. The diagonal of this square is made up of the diagonal of a square with a side length of 1, the altitude of an equilateral triangle with a side length of 2, the shortest altitude of a right triangle with a hypotenuse of length 2, and the diagonal of a square with a side length of 1. It follows that the diagonal of this square is $\sqrt{2}+\sqrt{3}+1+\sqrt{2}$, and therefore, the minimum value of s is $\boxed{\dfrac{4+\sqrt{2}+\sqrt{6}}{2}}$.

15. Consider triangle $\triangle ABC$ in the xy-plane where A is at the origin, B lies on the positive x-axis, C is on the upper right quadrant, and $\angle A = 30°$, $\angle B = 60°$, and $\angle C = 90°$. Let the length $\overline{BC} = 1$. Draw the angle bisector l of angle $\angle C$, and let this intersect the y-axis at D. What is the area of quadrilateral $ADBC$?



Solution: Let the foot of the perpendicular from C to $\overline{AD}$ be E. We know that $\overline{AC} = \sqrt{3}$ and $\overline{AB} = 2$. We also know that $\triangle AEC$ is a $30-60-90$ right triangle. It follows that $\overline{EC} = \frac{3}{2}$ and $\overline{EA} = \frac{\sqrt{3}}{2}$. We also know that $\angle ECD = 75°$. It follows that $\overline{ED} = \frac{3}{2}\cdot\tan 75° = \frac{3}{2}\cdot(2+\sqrt{3}) = \frac{6+3\sqrt{3}}{2}$.

It follows that $\overline{AD} = \frac{6+3\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 3+\sqrt{3}$. Therefore, the area of $\triangle ABD$ is $(3+\sqrt{3})\cdot 2\cdot \frac{1}{2} = 3+\sqrt{3}$. In addition, the area of $\triangle ABC$ is $\frac{\sqrt{3}}{2}$. Therefore our answer is $3+\sqrt{3}+\frac{\sqrt{3}}{2} = \boxed{\dfrac{6+3\sqrt{3}}{2}}$.

3 Sources

1. 2015 Berkeley Math Tournament Spring Individual Problem 3
2. 2015 Berkeley Math Tournament Spring Individual Problem 7
3. 2015 Berkeley Math Tournament Spring Geometry Problem 1
4. 2015 Berkeley Math Tournament Spring Geometry Problem 2
5. 2015 Berkeley Math Tournament Spring Geometry Problem 3
6. 2015 Berkeley Math Tournament Spring Geometry Problem 4
7. 2015 Berkeley Math Tournament Spring Geometry Problem 5
8. 2015 Berkeley Math Tournament Spring Team Problem 4
9. 2016 Berkeley Math Tournament Fall Individual Problem 12
10. 2016 Berkeley Math Tournament Fall Team Problem 11
11. 2016 Berkeley Math Tournament Fall Team Problem 12
12. 2016 Berkeley Math Tournament Fall Team Problem 19
13. 2016 Berkeley Math Tournament Spring Individual Problem 4
14. 2016 Berkeley Math Tournament Spring Individual Problem 14
15. 2016 Berkeley Math Tournament Spring Individual Problem 17