Geometry Handout #7 Answers and Solutions

Walker Kroubalkian

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1 Answers

1. 30

2. $5\sqrt{2}$

3. $\frac{5\sqrt{3}}{12}$

4. 110°

5. $\frac{3-\sqrt{3}}{2}$

6. 10

7. $\frac{1}{2}$

8. $\frac{2-\sqrt{3}}{2}$

9. $\sqrt{10}$

10. $\frac{9-5\sqrt{3}}{3}$

11. $\sqrt{41}$

12. $\frac{2}{3}$

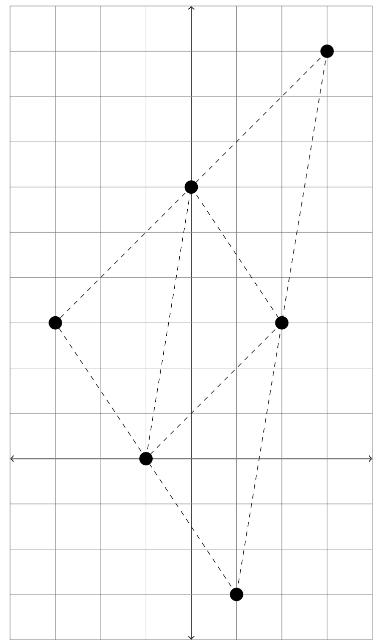
13. $\frac{1}{8}$

14. $\frac{5}{2}$

15. $\sqrt{2}-1$

2 Solutions

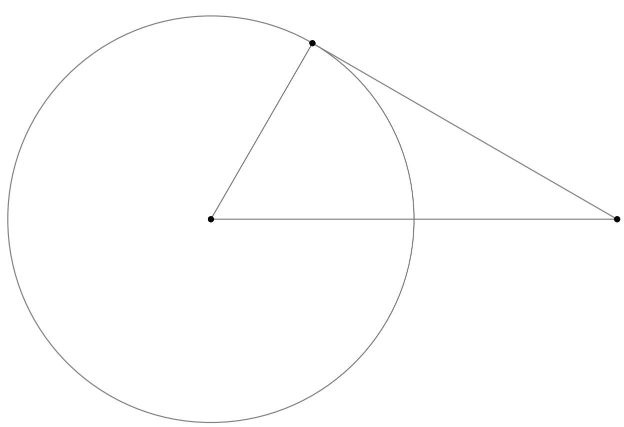
1. Let S be the set of points A in the Cartesian plane such that the four points A, $(2,3)$, $(-1,0)$, and $(0,6)$ form the vertices of a parallelogram. Let P be the convex polygon whose vertices are the points in S. What is the area of P?



Solution: We can recall that the two diagonals of any parallelogram must bisect each other. It follows that A must be the reflection of one of the points over the midpoint of the segment which connects the other two points. We can easily calculate that the midpoint of the segment which connects (a, b) and (c, d) is $(\frac{a+c}{2}, \frac{b+d}{2})$. Additionally, we can easily calculate that the reflection of the point (a, b) over the point (c, d) is the point $(2c - a, 2d - b)$. Using these facts, we can find that the three possible coordinates for A are $(3, 9), (-3, 3)$, and $(1, -3)$. Using the Shoelace Formula, it follows that the area of P is $\frac{\|3 \cdot 3 + (-3) \cdot (-3) + 1 \cdot 9 - 9 \cdot (-3) - 3 \cdot 1 - (-3) \cdot 3\|}{2} = \boxed{30}$.

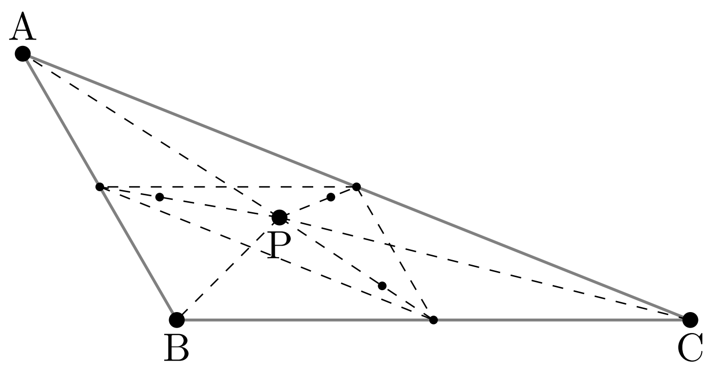
Alternatively, we could notice that P is made up of 4 triangles which are all congruent to the triangle connecting $(2, 3), (-1, 0)$, and $(0, 6)$, and therefore our answer is 4 times the area of that triangle.

2. Two points are located 10 units apart, and a circle is drawn with radius r centered at one of the points. A tangent line to the circle is drawn from the other point. What value of r maximizes the area of the triangle formed by the two points and the point of tangency?



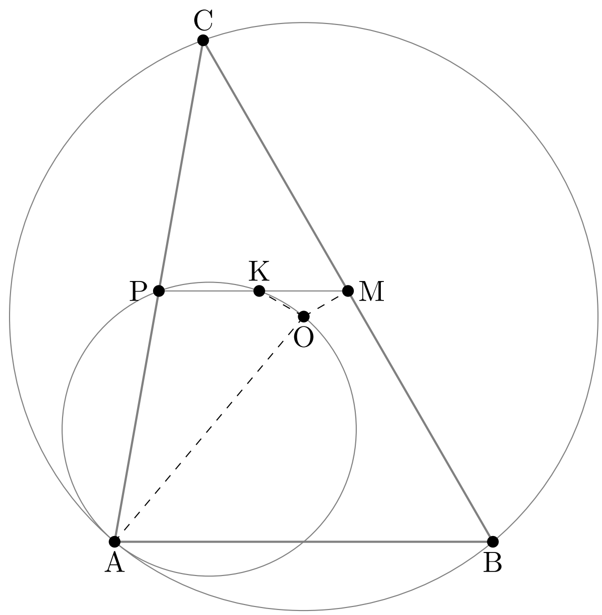
Solution: Let the angle between the radius and the line connecting the two points be θ. Then we have that the area of the triangle is $\frac{100\cos\theta\sin\theta}{2} = 25\sin(2\theta)$. It follows that the maximum area is 25 when $\sin(2\theta) = 1$. Therefore, we must have $\theta = \frac{\pi}{4}$, and it follows that the optimal radius is $r = 10\cos(\frac{\pi}{4}) = \boxed{5\sqrt{2}}$.

3. Let $\triangle ABC$ be a triangle with $\overline{AB} = 3$, $\overline{BC} = 5$, and $\overline{AC} = 7$, and let P be a point in its interior. If G_A, G_B, G_C are the centroids of $\triangle PBC, \triangle PAC, \triangle PAB$, respectively, find the maximum possible area of $\triangle G_A G_B G_C$.



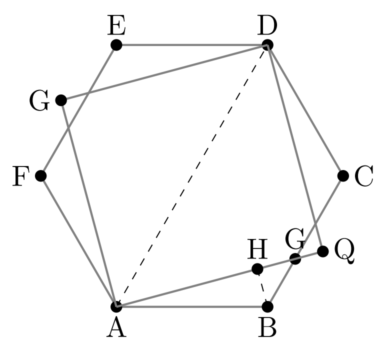
Solution: Let the midpoint of $\overline{BC}$ be A', and define B' and C' similarly. We know that $\frac{\overline{PG_A}}{\overline{PA'}} = \frac{\overline{PG_B}}{\overline{PB'}} = \frac{\overline{PG_C}}{\overline{PC'}} = \frac{2}{3}$. It follows that there is a homothety centered at P that translates $\triangle G_A G_B G_C$ to $\triangle A'B'C'$ with a scale factor of $\dfrac{2}{3}$. It follows that the area of $\triangle G_A G_B G_C$ is always $(\frac{2}{3})^2 = \frac{4}{9}$ of the area of $\triangle A'B'C'$. We know that the area of $\triangle A'B'C'$ is $\frac{1}{4}$ of the area of $\triangle ABC$. By Heron's Formula, the area of $\triangle ABC$ is $\sqrt{7.5 \cdot 0.5 \cdot 2.5 \cdot 4.5} = \frac{15\sqrt{3}}{4}$. Therefore, our answer is $\frac{1}{4} \cdot \frac{4}{9} \cdot \frac{15\sqrt{3}}{4} = \boxed{\dfrac{5\sqrt{3}}{12}}$.

4. Triangle $\triangle ABC$ is drawn such that $\angle A = 80°$, $\angle B = 60°$, and $\angle C = 40°$. Let the circumcenter of $\triangle ABC$ be O, and let ω be the circle with diameter $\overline{AO}$. Circle ω intersects side $\overline{AC}$ at point P. Let M be the midpoint of side $\overline{BC}$, and let the intersection of ω and $\overline{PM}$ be K. Find the measure of $\angle MOK$.



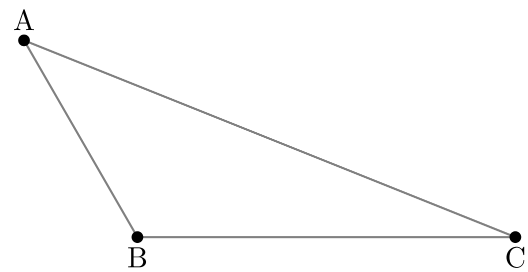
Solution: By definition, we know that $\angle OMC = 90°$, and by similar triangles, we know that $\angle PMC = 60°$. It follows that $\angle KMO = 90° - 60° = 30°$. In addition, because $OAPK$ is a cyclic quadrilateral, we know that $\angle PKO = 180° - \angle OAP = 180° - \frac{80°}{2} = 140°$, so it follows that $\angle MKO = 40°$. It follows that $\angle MOK = 180° - 40° - 30° = \boxed{110°}$.

5. Let $ABCDEF$ be a regular hexagon with side length 1. Now construct square $AGDQ$. What is the area of the region inside the hexagon and not the square?



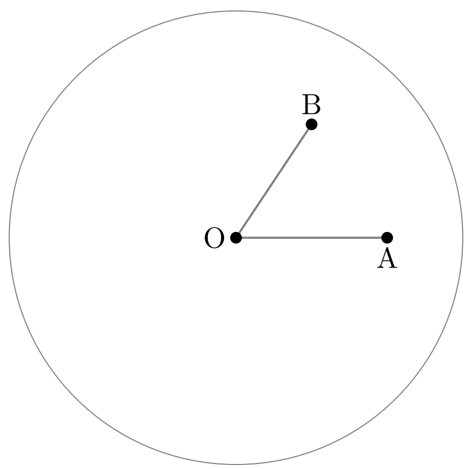
Solution: Notice that the area of the region inside the hexagon and not the square consists of 4 congruent triangles. Let $\overline{BC}$ intersect $\overline{AQ}$ at G as shown. We wish to calculate 4 times the area of $\triangle ABG$. We know that $\angle BAG = \angle BAD - \angle QAD = 60° - 45° = 15°$. Let H be the foot of the perpendicular from B to $\overline{AQ}$, as shown. We know that $\overline{AH} = \cos 15°$, and that $\angle ABH = 75°$. It follows that $\triangle BGH$ is an isosceles right triangle. Therefore, $\overline{BH} = \overline{GH} = \sin 15°$. It follows that $\overline{AG} = \sin 15° + \cos 15° = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{6}}{2}$. It follows that the area of $\triangle ABG$ is $\frac{\sqrt{6}}{2} \cdot 1 \cdot \frac{1}{2} \cdot \sin 15° = \frac{3-\sqrt{3}}{8}$. It follows that the area of the four triangles is $4 \cdot \frac{3-\sqrt{3}}{8} = \boxed{\frac{3-\sqrt{3}}{2}}$

6. Suppose the side lengths of triangle $\triangle ABC$ are the roots of the polynomial $x^3 - 27x^2 + 222x - 540$. What is the product of its inradius and circumradius?



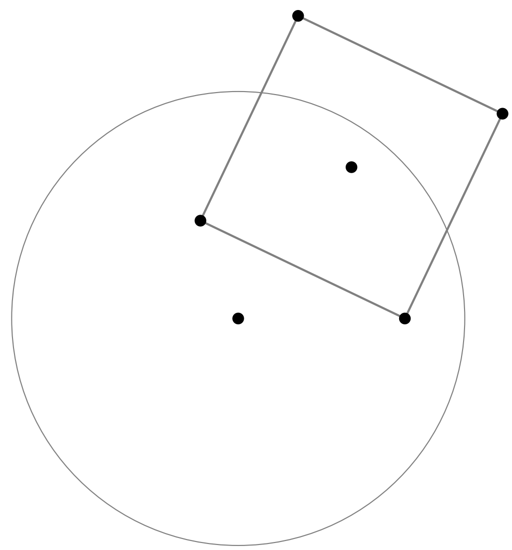
Solution: Let the sides of $\triangle ABC$ be a, b, and c. Let the inradius of $\triangle ABC$ be r, and let the circumradius of $\triangle ABC$ be R. Finally, let the area of $\triangle ABC$ be A. We know that $A = \frac{(a+b+c)r}{2} = \frac{abc}{4R}$. It follows that $rR = \frac{abc}{2a+2b+2c}$. By Vieta's formulas, we know that $a+b+c = 27$ and $abc = 540$. It follows that our answer is $\frac{540}{2 \cdot 27} = \boxed{10}$.

7. 2 darts are thrown randomly at a circular board with center O, such that each dart has an equal probability of hitting any point on the board. The points at which they land are marked A and B. What is the probability that $\angle AOB$ is acute?



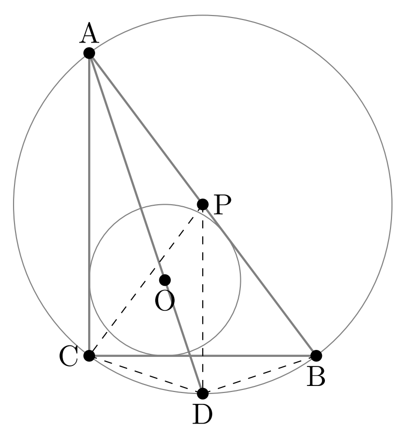
Solution: Consider an arbitrary point A and fix its location. We want to find the probability that for a randomly chosen point B, $\angle AOB < 90°$. It follows that if B is either $90°$ above or below $\overline{AO}$, then the condition will be satisfied. It follows that of all of the directions that B can be with respect to O, only $\frac{180°}{360°} = \boxed{\dfrac{1}{2}}$ of the directions will work.

8. The center of a square of side length 1 is placed uniformly at random inside a circle of radius 1. Given that we are allowed to rotate the square about its center, what is the probability that the entire square is contained within the circle for some orientation of the square?



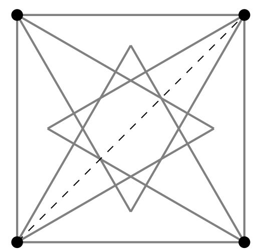
Solution: Notice that the square can be contained with the circle if and only if it is contained within the circle when the midpoint of one of the sides of the square is on the line connecting the center of the circle and the center of the square. In this orientation, the maximum distance from the center of the circle to one of the sides of the square is $\frac{\sqrt{3}}{2}$ as it is the height of an equilateral triangle with side length 1. This means that the distance between the center of the square and the center of the circle is at most $\frac{\sqrt{3}}{2} - \frac{1}{2}$. It follows that the probability that this orientation can be obtained is $(\frac{\frac{\sqrt{3}-1}{2}}{1})^2 = \boxed{\frac{2-\sqrt{3}}{2}}$.

9. Let $\overline{AB} = 10$ be a diameter of circle P. Pick point C on the circle such that $\overline{AC} = 8$. Let the circle with center O be the incircle of $\triangle ABC$. Extend line $\overline{AO}$ to intersect circle P again at D. Find the length of $\overline{BD}$.



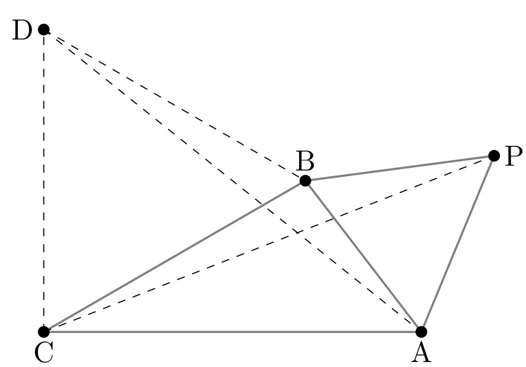
Solution: Because $\angle CAD \equiv \angle DAB$, we must have that $\overset{\frown}{CD} \cong \overset{\frown}{BD}$. It follows that $\overline{CD} \cong \overline{BD}$. Because $\overline{AB}$ is a diameter of circle P, we must have that $\angle ACB$ is a right angle. Therefore, by the Pythagorean Theorem, we must have that $\overline{CB} = \sqrt{10^2 - 8^2} = 6$. Similarly, if we let P be the circumcenter of $\triangle ABC$, we must have that $\overline{PC} = \overline{PB} = \overline{PD} = 5$. Because both $\triangle PCB$ and $\triangle BCD$ are isosceles triangles, we know that $\overline{PD}$ is a perpendicular bisector of $\overline{BC}$. If we let the intersection of $\overline{PD}$ and $\overline{BC}$ be E, then it follows that $\overline{EC} = \frac{6}{2} = 3$, and by the Pythagorean Theorem, it follows that $\overline{PE} = \sqrt{5^2 - 3^2} = 4$. It follows that $\overline{ED} = \overline{PD} - \overline{PE} = 1$. Therefore, by the Pythagorean Theorem, we know that $\overline{BD} = \sqrt{3^2 + 1^2} = \boxed{\sqrt{10}}$.

10. 4 equilateral triangles of side length 1 are drawn on the interior of a unit square, each one of which shares a side with one of the 4 sides of the unit square. What is the common area enclosed by all 4 equilateral triangles?



Solution: Notice that by symmetry, the common area must be an equilateral octagon with rotational 90° symmetry (equilateral meaning "all sides are equal"). We will proceed with coordinates. Let the bottom left corner have coordinates $(0, 0)$, let the bottom right corner have coordinates $(1, 0)$, let the top left corner have coordinates $(0, 1)$, and let the top right corner have coordinates $(1, 1)$. We can find through $30 - 60 - 90$ triangles that the bottom vertex of the equilateral octagon has coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{6})$. By symmetry, it follows that the top vertex of the equilateral octagon has coordinates $(\frac{1}{2}, 1 - \frac{\sqrt{3}}{6})$, and therefore, one of the diagonals of the octagon has length $1 - \frac{\sqrt{3}}{3}$. We can find through trigonometry that the other diagonal of the octagon has length $2 \cdot \frac{\sqrt{2}}{2} \cdot \tan 15° = 2\sqrt{2} - \sqrt{6}$. We know that the area of an octagon with this kind of symmetry is $8 \cdot \frac{p}{2} \cdot \frac{q}{2} \cdot \frac{1}{2} \cdot \sin 45° = \frac{pq\sqrt{2}}{2}$ where the diagonals are p and q. It follows that our answer is $(2\sqrt{2} - \sqrt{6}) \cdot (\frac{3-\sqrt{3}}{3}) \cdot \frac{\sqrt{2}}{2} = \boxed{\dfrac{9 - 5\sqrt{3}}{3}}$.

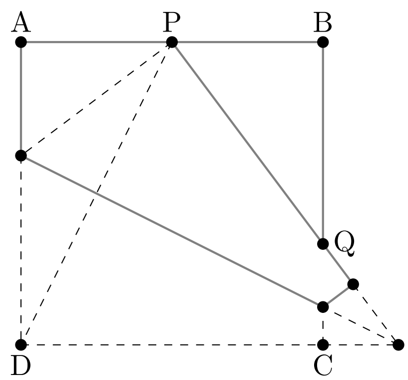
11. In triangle $\triangle ABC$, the angle at C is 30°, side $\overline{BC}$ has length 4, and side $\overline{AC}$ has length 5. Let P be the point such that triangle $\triangle ABP$ is equilateral and non-overlapping with triangle $\triangle ABC$. Find the distance from C to P.



Solution: Construct equilateral triangle $\triangle BCD$ on side $\overline{BC}$ of $\triangle ABC$. Notice that by definition, $\overline{BA} = \overline{BP}$. In addition, we know that $\overline{BC} = \overline{BD}$. Finally, if we let $\angle CBA = x$, then we have that

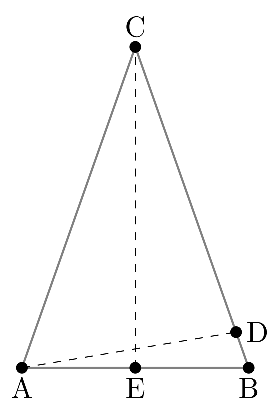
$\angle ABD \cong \angle PBC = x + 60°$. It follows by SAS that $\triangle DBA$ is congruent to $\triangle CBP$. Therefore, $\overline{CP} = \overline{DA}$. Finally, we know that $\angle DCA = \angle DCB + \angle BCA = 60° + 30° = 90°$. It follows by the Pythagorean Theorem that $\overline{CP} = \overline{DA} = \sqrt{4^2 + 5^2} = \boxed{\sqrt{41}}$.

12. Take a square $ABCD$ of side length 1, and let P be the midpoint of $\overline{AB}$. Fold the square so that point D touches P, and let the intersection of the bottom edge $\overline{DC}$ with the right edge be Q. What is $\overline{BQ}$?



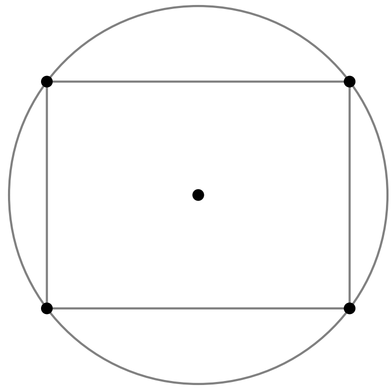
Solution: We will proceed with coordinates. Let D have coordinates $(0, 0)$, let C have coordinates $(1, 0)$, let B have coordinates $(1, 1)$, and let A have coordinates $(0, 1)$. We know that the fold is the perpendicular bisector of $\overline{PD}$. It follows that the equation for the fold is $y = -\frac{1}{2}x + \frac{5}{8}$. We know that the intersection of this line with the extension of line $\overline{CD}$ must lie on the extension of $\overline{PQ}$. We know that this will occur at $x = \frac{5}{4}$. Therefore, we know that line $\overline{PQ}$ passes through $(\frac{1}{2}, 1)$ and $(\frac{5}{4}, 0)$. It follows that the equation of line $\overline{PQ}$ is $y = -\frac{4}{3}x + \frac{5}{3}$. When $x = 1$, $y = \frac{1}{3}$, so it follows that $\overline{BQ} = 1 - \frac{1}{3} = \boxed{\dfrac{2}{3}}$.

13. Let $\triangle ABC$ be a triangle with $\overline{AB} = 1$, $\overline{AC} = 3$, and $\overline{BC} = 3$. Let D be a point on $\overline{BC}$ such that $\overline{BD} = \frac{1}{3}$. What is the ratio of the area of $\triangle BAD$ to the area of $\triangle CAD$?



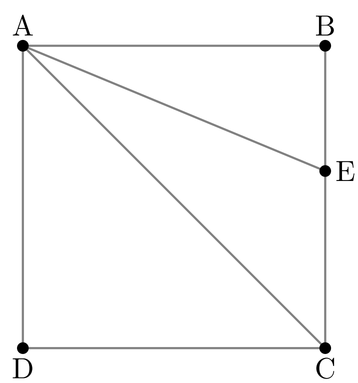
Solution: Because the altitude from A to $\overline{CD}$ has the same length as the altitude from A to $\overline{BD}$ as both are equal to the altitude from A to $\overline{BC}$, we must have that the ratio of the area of $\triangle BAD$ to the area of $\triangle CAD$ is equivalent to the ratio of their bases, or $\frac{\overline{BD}}{\overline{CD}} = \dfrac{\frac{1}{3}}{3 - \frac{1}{3}} = \boxed{\dfrac{1}{8}}$.

14. If a parallelogram with perimeter 14 and area 12 is inscribed in a circle, what is the radius of the circle?



Solution: We can recall that if one of the angles in a parallelogram is α, then the angle opposite the first angle in the parallelogram is also α. However, in a cyclic quadrilateral, we know that if one of the angles is α, then the angle opposite the first angle in the cyclic quadrilateral is $\pi - \alpha$. It follows that for any angle α in a cyclic parallelogram, we must have that $\alpha = \pi - \alpha$, or $\alpha = \frac{\pi}{2}$. It follows that any cyclic parallelogram must be a rectangle. Let the length of the rectangle be x and let the width of the rectangle be y. We know that $2x + 2y = 14$ and that $xy = 12$. It follows that if $x = 3$ and $y = 4$, then everything works. This would mean that the diameter of the circle or the diagonal of the parallelogram is 5, and it follows that the radius of the circle is $\boxed{\frac{5}{2}}$.

15. Take a square $ABCD$ of side length 1, and draw $\overline{AC}$. Point E lies on $\overline{BC}$ such that $\overline{AE}$ bisects $\angle BAC$. What is the length of $\overline{BE}$?



Solution: By the Angle Bisector Theorem, we know that $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{BE}}{\overline{EC}}$. It follows that $\frac{\overline{BE}}{\overline{EC}} = \frac{1}{\sqrt{2}}$. It follows that $\frac{\overline{BE}}{\overline{BC}} = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$. It follows that our answer is $\boxed{\sqrt{2} - 1}$.

3 Sources

1. 2017 Berkeley Math Tournament Spring Individual Problem 10
2. 2017 Berkeley Math Tournament Spring Individual Problem 13

3. 2017 Berkeley Math Tournament Spring Individual Problem 16
4. 2017 Berkeley Math Tournament Spring Individual Problem 17
5. 2017 Berkeley Math Tournament Spring Geometry Problem 3
6. 2017 Berkeley Math Tournament Spring Geometry Problem 5
7. 2017 Berkeley Math Tournament Spring Team Problem 4
8. 2017 Berkeley Math Tournament Spring Team Problem 6
9. 2017 Berkeley Math Tournament Spring Team Problem 9
10. 2017 Berkeley Math Tournament Spring Team Problem 13
11. 2017 Berkeley Math Tournament Spring Team Problem 15
12. 2017 Berkeley Math Tournament Fall Individual Problem 16
13. 2017 Berkeley Math Tournament Fall Team Problem 4
14. 2017 Berkeley Math Tournament Fall Team Problem 14
15. 2017 Berkeley Math Tournament Fall Team Problem 15