Geometry Handout 1 Answers and Solutions

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1 Answers

1. $12\sqrt{5}$

2. $\frac{3}{8}$ or 0.375

3. $\frac{25\pi-50}{2}$ or $\frac{25\pi}{2}-25$

4. $\sqrt{13}$

5. $2-\sqrt{3}$

6. $\frac{14}{5}$ or 2.8

7. $\frac{47}{7}$

8. $\frac{\pi-3\sqrt{3}+3}{3}$ or $\frac{\pi}{3}+1-\sqrt{3}$ or equivalent values

9. $72\sqrt{3}$

10. $\frac{3\pi}{10}$

11. 252

12. $18\sqrt{5}$

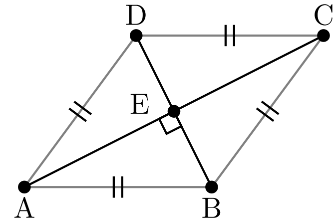
13. 13

14. 18

15. $\frac{180900}{1007}$

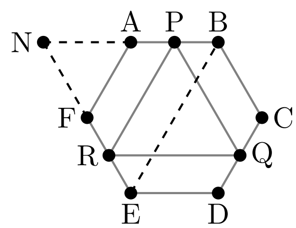
2 Solutions

1. A rhombus has area 36 and the longer diagonal is twice as long as the shorter diagonal. What is the perimeter of the rhombus?



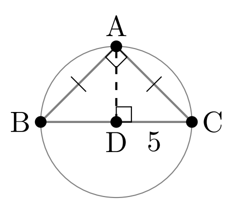
Solution: Label the points of the rhombus as shown above. By the properties of a rhombus, we have the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular bisectors of each other. Let $\overline{AC} = 2x$ and let $\overline{BD} = x$. Then we have the area of the rhombus is $\frac{AC \times BD}{2} = x^2 = 36$. Thus, $x = \overline{BD} = 6$. Then we have that $\overline{AE} = 6$ and $\overline{BE} = 3$. By the Pythagorean Theorem, we have $\overline{AB} = \sqrt{6^2 + 3^2} = 3\sqrt{5}$. Therefore, the perimeter of $ABCD$ is $4 \times 3\sqrt{5} = \boxed{12\sqrt{5}}$ as desired.

2. Given regular hexagon $ABCDEF$, compute the probability that a randomly chosen point inside the hexagon is inside triangle PQR, where P is the midpoint of $\overline{AB}$, Q is the midpoint of $\overline{CD}$, and R is the midpoint of $\overline{EF}$.



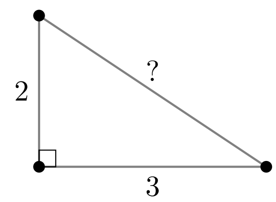
Extend lines EF and AB to intersect at N as shown above. Because $\angle EFA \cong \angle BAF = 120°$, we have $\angle NAF \cong \angle NFA = 180 - 120 = 60°$. Therefore, $\triangle NAF$ is an equilateral triangle, $\overline{NA} \cong \overline{NF}$, and $\angle ANF = 60°$. Because $\overline{PA} \cong \overline{RF} = \frac{\overline{AB}}{2}$ and $\overline{NA} \cong \overline{NF}$, we have $\overline{NR} \cong \overline{NP} = \overline{NA} + \overline{AP} = \overline{NF} + \overline{FR}$. Because $\angle RNP = 60°$ and $\overline{NR} = \overline{NP}$, we have $\triangle PNR$ is also equilateral. Thus, if we let $\overline{AP} = x$, then we have $\overline{PR} \cong \overline{NP} = 3x$. The probability that a randomly chosen point will land in $\triangle PQR$ is the same as the ratio of the area of $\triangle PQR$ to the area of $ABCDEF$. The area of $\triangle PQR$ is $\frac{s^2\sqrt{3}}{4} = \frac{(3x)^2\sqrt{3}}{4} = \frac{9x^2\sqrt{3}}{4}$ and the area of $ABCDEF$ is $\frac{3s^2\sqrt{3}}{2} = \frac{3(2x)^2\sqrt{3}}{2} = 6x^2\sqrt{3}$. Therefore our answer is $\frac{\frac{9x^2\sqrt{3}}{4}}{6x^2\sqrt{3}} = \boxed{\frac{3}{8} = 0.375}$ as desired.

3. An isosceles right triangle is inscribed in a circle of radius 5, thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.



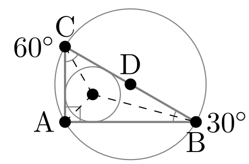
Solution: Label the diagram as shown above. By symmetry, we have that $\angle ADC = 90°$, where D is the center of the circle and the midpoint of the hypotenuse of the right isosceles triangle. We wish to find the sums of the areas of the two sectors from $\angle ADB$ and $\angle ADC$ which are not included in $\triangle ABC$. Therefore, our answer is $\frac{\pi r^2}{2} - \frac{bh}{2} = \frac{25\pi}{2} - \frac{10*5}{2} = \boxed{\frac{25\pi - 50}{2} = \frac{25\pi}{2} - 25}$.

4. A triangle with side lengths 2 and 3 has an area of 3. Compute the third side length of the triangle.



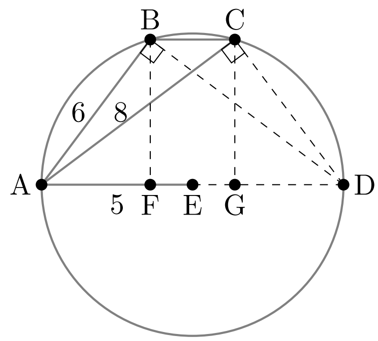
Solution: If the triangle has an area of 3 and a base of 3, then by $A = \frac{bh}{2}$, we have that the height to the side of length 3 is 2. Because this is equal to another side length of the triangle, we have that there must be a right angle between the side of length 2 and the side of length 3. Thus, by the Pythagorean Theorem, the third side length is $\sqrt{2^2+3^2} = \boxed{\sqrt{13}}$.

5. Compute the square of the distance between the incenter (center of the inscribed circle) and circumcenter (center of the circumscribed circle) of a 30-60-90 triangle with hypotenuse of length 2.



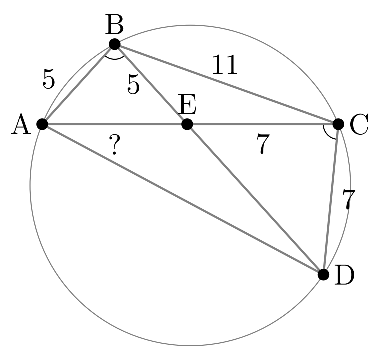
Solution: Label the diagram as shown above. Because the triangle is a right triangle, we have that the center of the circumscribed circle (the circumcenter) is at the midpoint of the hypotenuse of the triangle. This is because if a circle was circumscribed about a right triangle, the 90^{circ} angle would subtend an arc of $2 * 90^{circ} = 180°$, therefore subtending half of the circle. It follows that the hypotenuse is a diameter, and therefore its midpoint is the circumcenter, or point D in our diagram. We will proceed with coordinates. Let A be at $(0, 0)$ in the coordinate plane. By the nature of $30-60-90$ triangles, we have that $\frac{\overline{AC}}{\overline{BC}} = \frac{1}{2}$, meaning $\overline{AC} = 2 \times \frac{1}{2} = 1$. Therefore, we will let C be at point $(0, 1)$ in the coordinate plane. Similarly, $\frac{\overline{AB}}{\overline{BC}} = \frac{\sqrt{3}}{2}$. Therefore, $\overline{AB} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$. Therefore, we will let B be at point $(\sqrt{3}, 0)$ in the coordinate plane. Because D is at the midpoint of $\overline{BC}$, we have that the circumcenter D is at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ in the coordinate plane. Now, if we let the radius of the inscribed circle be r, and the center of the inscribed circle be I, then by area we have tha $|\triangle ABC| = |\triangle ABI| + |\triangle BCI| + |\triangle ACI|$ where $|\triangle ABC|$ represents the area of $\triangle ABC$ and so on. By $A = \frac{bh}{2}$, we have $\frac{\sqrt{3}}{2} = \frac{r}{2} \times (1 + \sqrt{3} + 2)$. Solving this equation we get $r = \frac{\sqrt{3}-1}{2}$. We have that I is at $(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2})$ in the coordinate plane. Thus, we wish to calculate the square of the distance between $(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2})$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. By the distance formula, we get that this is equal to $(\frac{1}{2})^2 + (\frac{\sqrt{3}-2}{2})^2 = \boxed{2-\sqrt{3}}$

6. Points A, B, and C lie on a circle of radius 5 such that $\overline{AB} = 6$ and $\overline{AC} = 8$. Find the smaller of the two possible values of $\overline{BC}$.

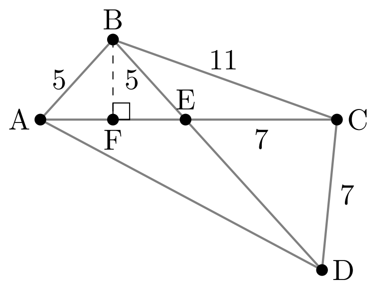


Solution: Let D be the point diametrically opposite of point A on the circle. Consider $\angle ACD$ and $\angle ABD$. Both of these angles subtend half of the circle, and therefore both are right angles. By the Pythagorean Theorem, it follows that $\overline{BD} = \sqrt{10^2 - 6^2} = 8$ and $\overline{CD} = \sqrt{10^2 - 8^2} = 6$. Drop a perpendicular from point B to $\overline{AD}$ and call the foot F. Similarly, drop a perpendicular from point C to $\overline{AD}$ and call the foot G. We have that $\overline{BC} \cong \overline{FG}$. By $A = \frac{bh}{2}$, we have that $\overline{BF} = \frac{6\times 8}{10} = \frac{24}{5}$. By the Pythagorean Theorem, we have that $\overline{AF} = \sqrt{6^2 - (\frac{24}{5})^2} = \frac{18}{5}$. It follows that $\overline{EF} = 5 - \frac{18}{5} = \frac{7}{5}$, and therefore $\overline{FG} \cong \overline{BC} = \frac{2\times 7}{5} = \boxed{\frac{14}{5}}$ by symmetry.

7. In quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. If $\overline{AB} \cong \overline{BE} = 5$, $\overline{EC} \cong \overline{CD} = 7$, and $\overline{BC} = 11$, compute $\overline{AE}$.

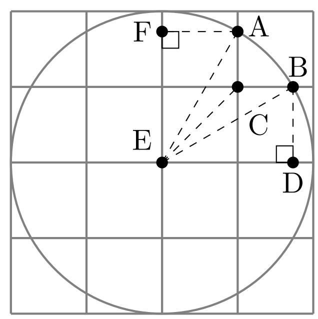


Solution 1: Notice that $\angle AEB \cong \angle CED$ and both $\triangle AEB$ and $\triangle ECD$ are isosceles. It follows that $\angle BAE \cong \angle AEB \cong \angle CED \cong \angle EDC$, and therefore by AA similarity, $\triangle AEB$ and $\triangle EDC$ are similar. It follows that $\angle ACD \cong \angle ABD$. This is one of the conditions for a cyclic quadrilateral, and therefore $ABCD$ is cyclic. It follows by Power of a Point that if $\overline{AE} = 5x$, then $\overline{ED} = 7x$, and it follows by similarity that $\overline{AD} = 11x$. From here, we can use Ptolemy's Theorem to find that $(7x + 5) \times (5x + 7) = 11 \times 11x + 35$, and simplifying we get $47x = 35x^2$. This equation has two solutions, $x = 0$ and $x = \frac{47}{35}$. Clearly x is not 0, so our answer is $5 \times \frac{47}{35} = \boxed{\frac{47}{7}}$.



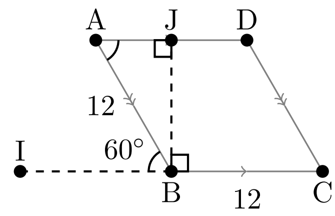
Solution 2: Let $\overline{AE} = 2x$. Let the foot of the perpendicular from B to $\overline{AC}$ be F. It follows by symmetry that $\overline{AF} \cong \overline{FE}$ and $\angle AFB \cong \angle CFB = 90°$. Let $\overline{AF} = x$, and let $\overline{BF} = h$. It follows by the Pythagorean Theorem that $h^2 = 25 - x^2 = 121 - (7+x)^2$. Solving this, we get $14x = 47$, or $x = \frac{47}{14}$. It follows that $2x = \overline{AE} = \boxed{\dfrac{47}{7}}$.

8. Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?



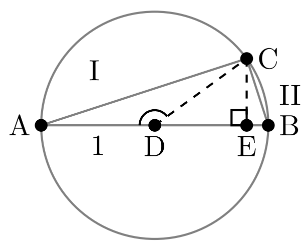
Solution: Label the diagram as shown above. We wish to calculate the piece of pizza enclosed by the points A, B, and C. Notice that this area can be thought of as taking the area of the sector enclosed by arc $\overset{\frown}{AB}$ and subtracting the areas of $\triangle EAC$ and $\triangle ECB$. Let the foot of the perpendicular from B to the horizontal radius be D as shown. It follows that $\overline{EB} = 2$ and $\overline{BD} = 1$. Because $\frac{\overline{BD}}{\overline{EB}} = \frac{1}{2}$, we have that $\angle EDB = 30°$ as $\triangle EDB$ is a $30 - 60 - 90$ triangle. It follows that $\overline{ED} = \sqrt{3}$. It follows that $\overline{BC} \cong \overline{AC} = \sqrt{3} - 1$ by symmetry. Using $A = \frac{bh}{2}$, we have that $|\triangle EAC| = |\triangle EBD| = \frac{\sqrt{3}-1}{2}$ where $|\triangle EAC|$ is the area of $\triangle EAC$ and so on. Now consider the foot of the perpendicular from A to the vertical radius as shown, and call this foot F. By symmetry, we have $\angle FEA \cong \angle BED = 30°$. It follows that $\angle AEB = 90 - 30 - 30 = 30°$. It follows that the area of the sector enclosed by $\overset{\frown}{AB}$ is $2^2 \times \pi \times \frac{30°}{360°} = \frac{\pi}{3}$. Therefore, our answer is $\boxed{\dfrac{\pi - 3\sqrt{3} + 3}{3} \text{ or } \dfrac{\pi}{3} + 1 - \sqrt{3}}$ as desired.

9. $ABCD$ is a parallelogram. $\overline{AB} \cong \overline{BC} = 12$, and $\angle ABC = 120°$. Calculate the area of parallelogram $ABCD$.



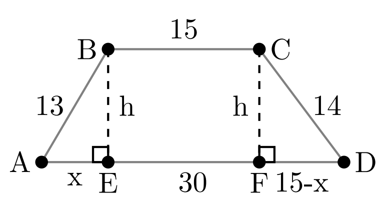
Solution: Extend $\overline{BC}$ past B to some point I. Notice that $\angle ABI = 180 - 120 = 60°$. It follows because $\overline{AD}||\overline{BC}$ that $\angle BAD = 60°$. Drop a perpendicular from B to $\overline{AD}$ and call the foot J as shown. It follows that $\triangle AJB$ is a $30-60-90$ triangle. Therefore, $\overline{BJ} = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$. Because the area of a parallelogram is $A = bh$, it follows that the area of parallelogram $ABCD$ is $12 \times 6\sqrt{3} = \boxed{72\sqrt{3}}$.

10. A circle with radius 1 has diameter $\overline{AB}$. C lies on this circle such that the ratio of arc $\overline{AC}$ to arc $\overline{BC}$ is 4. Segment $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, segment $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.



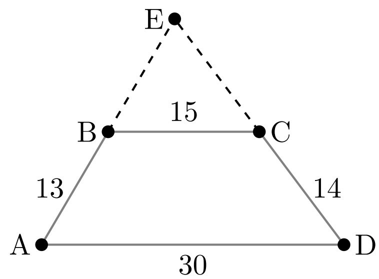
Solution: Notice that region I is the area in the sector bounded by minor arc $\overset{\frown}{AC}$ and not inside $\triangle ADC$. Similarly, region II is the area in the sector bounded by minor arc $\overset{\frown}{BC}$ and not inside $\triangle BDC$. Therefore, we wish to calculate $|\overset{\frown}{AC}| + |\triangle ADC| - |\overset{\frown}{BC}| - |\triangle BDC|$ where $|\overset{\frown}{AC}|$ is the area of the sector bounded by minor arc $\overset{\frown}{AC}$ and so on. Notice that because $\overline{AD} \cong \overline{BD}$, and because $\triangle ADC$ and $\triangle BDC$ share the same height to these respective segments ($\overline{CE}$), by $A = \frac{bh}{2}$, we have that $|\triangle ADC| = |\triangle BDC|$. Thus, we just wish to compute $|\overset{\frown}{AC}| - |\overset{\frown}{BC}|$. Because $\angle ADC = 4 \times \angle BDC$, and $\angle ADC + \angle BDC = 180°$, we have that $\angle ADC = 144°$ and $\angle BDC = 36°$. Thus, our result is $\pi \times (1)^2 \times \frac{144}{360} - \pi \times (1)^2 \times \frac{36}{360} = \boxed{\frac{3\pi}{10}}$ as desired.

11. In trapezoid $ABCD$, $\overline{BC}$ is parallel to $\overline{AD}$, $\overline{AB} = 13$, $\overline{BC} = 15$, $\overline{CD} = 14$, and $\overline{AD} = 30$. Find the area of $ABCD$.



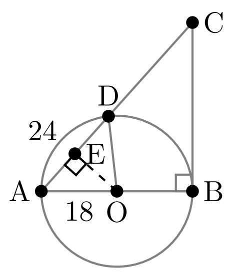
Solution 1: Drop perpendiculars from B and C to $\overline{AD}$ and call the feet E and F as shown above. Let $\overline{AE} = x$. It follows that $\overline{FD} = 30-15-x = 15-x$. Let $\overline{BE} = h$. It follows by the Pythagorean

Theorem that $h^2 = 13^2 - x^2 = 14^2 - (15-x)^2$. Solving gives us $x = \frac{198}{30} = \frac{33}{5}$. Plugging this back in gives us $h = \frac{56}{5}$. Because $A = \frac{(b_1+b_2)\times h}{2}$, our answer is $\frac{45\times 56}{2\times 5} = \boxed{252}$ as desired.



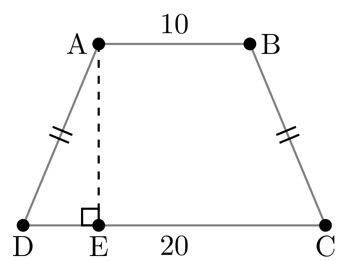
Solution 2: Extend $\overline{AB}$ past B and $\overline{CD}$ past C such that they intersect at point E as shown. Notice that because $\triangle EBC$ and $\triangle EAD$ share $\angle BEC \cong \angle AED$, and $\overline{BC}||\overline{AD}$, it follows that $\angle EBC \cong \angle EAD$, and by AA similarity we have $\triangle EBC \sim \triangle EAD$. It follows that $\overline{EB} = 13$ and $\overline{EC} = 14$. This means $\triangle EBC$ is a $13-14-15$ triangle. Because $\frac{\overline{AD}}{\overline{BC}} = 2$, it follows that $\frac{|\triangle EAD|}{|\triangle EBC|} = 2^2 = 4$. Either by memory, or by Heron's formula, we can find that $|\triangle EBC| = \sqrt{21\times(21-13)\times(21-14)\times(21-15)} = 84$. It follows that $|\triangle EAD| = 4\times 84 = 336$, and from here we can get that $|ABCD| = |\triangle EAD| - |\triangle EBC| = 336 - 84 = \boxed{252}$ as desired.

12. Circle O has radius 18. From diameter $\overline{AB}$, there exists a point C such that $\overline{BC}$ is tangent to O and $\overline{AC}$ intersects O at a point D, with $\overline{AD} = 24$. What is the length of $\overline{BC}$?



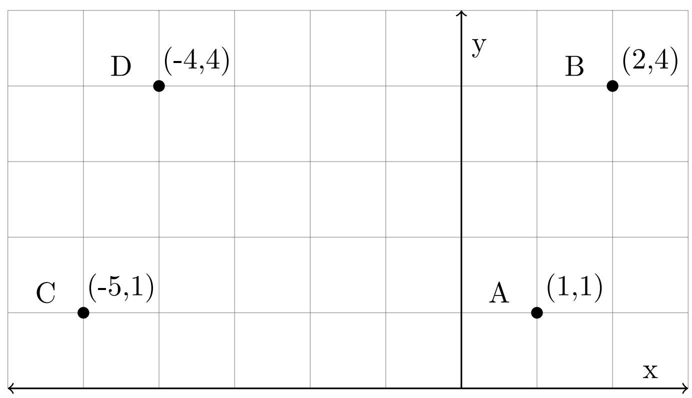
Solution: Drop a perpendicular from O to $\overline{AD}$ and call the foot E as shown. Because $\overline{AO} \cong \overline{OD}$, $\triangle AOD$ is isosceles and therefore $\overline{AE} \cong \overline{ED}$. It follows that $\overline{AE} = \frac{24}{2} = 12$. It follows by the Pythagorean Theorem that $\overline{OE} = \sqrt{18^2 - 12^2} = 6\sqrt{5}$. Notice that $\triangle AEO$ and $\triangle ABC$ share $\angle EAO \cong \angle BAC$. They also have $\angle AEO \cong \angle ABC = 90°$. It follows by AA similarity that $\triangle AEO \sim \triangle ABC$. As a result, $\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{AE}}{\overline{EO}}$. It follows that $\overline{BC} = \frac{\overline{AB}\times\overline{EO}}{\overline{AE}} = \frac{36\times 6\sqrt{5}}{12} = \boxed{18\sqrt{5}}$ as desired.

13. Isosceles trapezoid $ABCD$ has $\overline{AB} = 10$, $\overline{CD} = 20$, $\overline{BC} \cong \overline{AD}$, and an area of 180. Compute the length of $\overline{BC}$.

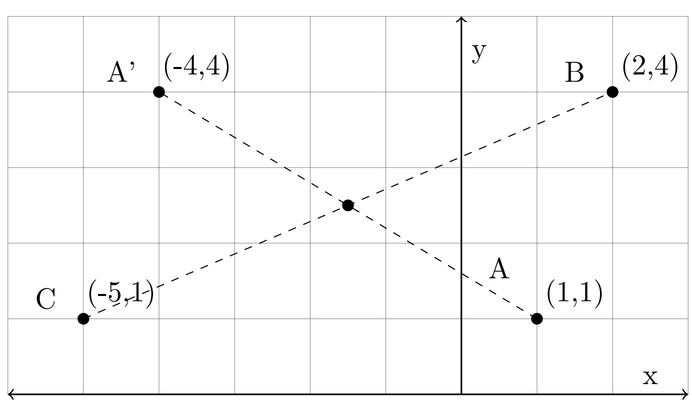


Solution: Drop a perpendicular from $\overline{AE}$ to $\overline{CD}$ as shown and call the foot E. By symmetry, we have that $\overline{ED} = \frac{20-10}{2} = 5$. By $A = \frac{(b_1+b_2)\times h}{2}$, it follows that $h = \frac{180\times 2}{10+20} = 12$. Therefore, $\overline{AE} = 12$. It follows by the Pythagorean Theorem that $\overline{AD} \cong \overline{BC} = \sqrt{12^2+5^2} = \boxed{13}$ as desired.

14. The coordinates of three vertices of a parallelogram are A $(1,1)$, B $(2,4)$, and C $(-5,1)$. Compute the area of the parallelogram.



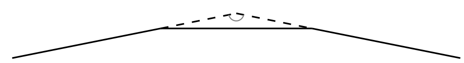
Solution 1: By inspection, we can find that any of the points $(-4,4)$, $(8,4)$, or $(-6,-2)$ work as a fourth point D to make $ABCD$ a parallelogram. Thus, because $A = b \times h$, we have that our answer is $6 \times 3 = \boxed{18}$.



Solution 2: Notice that in order to form a fourth point for our parallelogram, we must reflect one of the three points over the midpoint of the segment formed by the other two points. For example, in the diagram above, we reflect A over the midpoint of $\overline{BC}$ to get a fourth point A' to complete our parallelogram. This is equivalent to reflecting $\triangle ABC$ over one of its sides, or in the example

above, over $\overline{BC}$. Therefore, no matter which parallelogram is formed, the total area will always be double the area of $\triangle ABC$. Therefore, by $A = \frac{bh}{2}$, our answer is $2 \times \frac{6\times3}{2} = \boxed{18}$ as desired.

15. Given a regular 2014-gon, we construct 2014 isosceles triangles on the exterior of the polygon such that each isosceles triangle has an edge of the polygon as its base and has legs formed by the extensions of the two adjacent sides. Compute in degrees the largest angle of one such triangle. (Note: This figure is not drawn to scale.)



Solution: We either remember or notice that the sum of the exterior angles going in a clockwise or counter-clockwise direction around a polygon will be 360°. A simple proof for convex polygons such as regular polygons involves taking a marker and going around the perimeter of the polygon, lying the marker in the direction of each side. As you go in one direction, the marker makes one complete revolution, so it follows that the sum of the exterior angles in that direction must be 360°. In a regular 2014-gon, all 2014 exterior angles must be the same, so it follows that in each isosceles triangle, each of the base angles must be $\frac{360}{2014} = \frac{180}{1007}$. Therefore, our answer is $180 - \frac{2\times180}{1007} = \boxed{\frac{180900}{1007}}$ as desired.

3 Sources

1. 2013 Stanford Math Tournament General Problem 5
2. 2013 Stanford Math Tournament General Problem 15
3. 2013 Stanford Math Tournament General Problem 17
4. 2013 Stanford Math Tournament General Problem 19
5. 2013 Stanford Math Tournament General Problem 24
6. 2013 Stanford Math Tournament Geometry Tiebreaker Problem 2
7. 2013 Stanford Math Tournament Geometry Tiebreaker Problem 3
8. 2013 Stanford Math Tournament Geometry Problem 3
9. 2012 Stanford Math Tournament General Problem 15
10. 2012 Stanford Math Tournament General Problem 17
11. 2012 Stanford Math Tournament General Problem 21
12. 2012 Stanford Math Tournament General Problem 23
13. 2014 Stanford Math Tournament General Problem 7
14. 2014 Stanford Math Tournament General Problem 10
15. 2014 Stanford Math Tournament General Problem 15