Geometry Handout #4 Answers and Solutions

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1 Answers

1. 4π

2. $4 - 2\sqrt{3}$

3. $24\sqrt{3}$

4. 498

5. $\frac{16\sqrt{3}}{3}$

6. 14

7. $\frac{4\pi - 3\sqrt{3}}{18}$

8. $6\sqrt{5}$

9. $\sqrt{2}$

10. $4\sqrt{3}$

11. 32

12. $27\sqrt{3}$

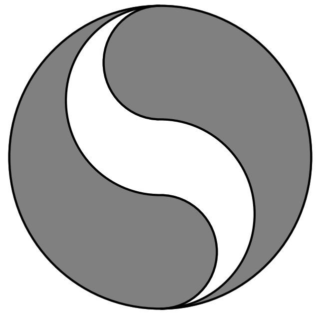
13. $\frac{1}{4}$

14. $\frac{18 + 3\sqrt{3}}{2}$

15. $\frac{4\pi - 3\sqrt{3}}{12}$

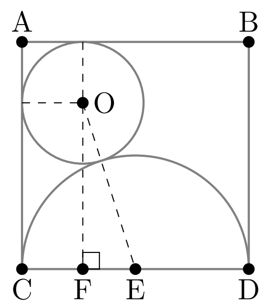
2 Solutions

1. S-Corporation designs its logo by linking together 4 semicircles along the diameter of a unit circle. Find the perimeter of the shaded portion of the logo.



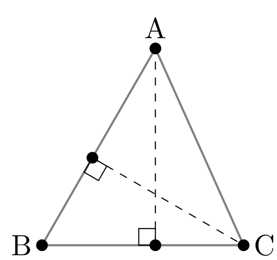
Solution: Notice that the perimeter of the shaded region is the same as the circumference of the unit circle added to the circumference of the small shaded circle and the circumference of the white circle. If we let the diameter of the shaded circle be s, then we have that the diameter of the white circle is $2 - s$. Therefore, our answer is $(2 + s + (2 - s))\pi = \boxed{4\pi}$.

2. Let $ABCD$ be a square with side length 2, and let a semicircle with flat side CD be drawn inside the square. Of the remaining area inside the square outside the semi-circle, the largest circle is drawn. What is the radius of this circle?



Solution: Label the diagram as shown above. Let the radius of circle O be r. Notice that the semicircle centered at E has a radius of 1, and therefore, $\overline{FE} = 1 - r$ and $\overline{OE} = 1 + r$. Finally, $\overline{OF} = 2-r$. By the Pythagorean Theorem, it follows that $(1-r)^2+(2-r)^2 = (1+r)^2$. Rearranging, we get $r^2 - 8r + 4 = 0$. Solving, we get that $r = 4 \pm 2\sqrt{3}$. Because r is clearly less, than 1, it follows that the largest possible radius is $\boxed{4 - 2\sqrt{3}}$.

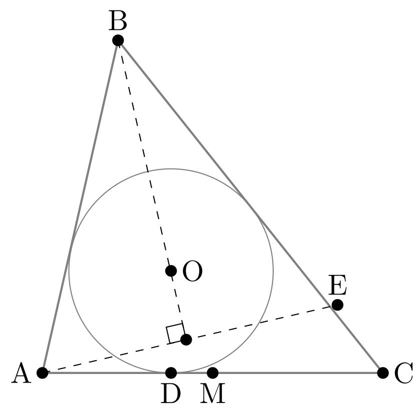
3. Triangle ABC satisfies the property that $\angle A = a \log x$, $\angle B = a \log 2x$, and $\angle C = a \log 4x$ radians, for some real numbers a and x. If the altitude to side $\overline{AB}$ has length 8 and the altitude to side $\overline{BC}$ has length 9, find the area of $\triangle ABC$.



Solution: Notice that $\angle B = a \log 2 + \angle A$ and $\angle C = a \log 2 + \angle B$. It follows that the three angles of $\triangle ABC$ form an arithmetic sequence, and therefore, the sum of its angles is $3\angle B$. It follows that

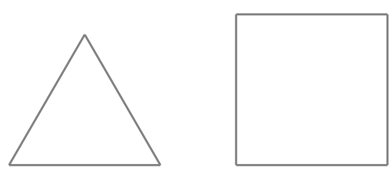
$\angle ABC = 60°$. Using the properties of $30-60-90$ right triangles, it follows that $\overline{BC} = \frac{2}{\sqrt{3}} \cdot 8 = \frac{16\sqrt{3}}{3}$ and that the area of $\triangle ABC$ is $\frac{48\sqrt{3}}{2} = \boxed{24\sqrt{3}}$ as desired.

4. Triangle $\triangle ABC$ has incircle O that is tangent to $\overline{AC}$ at D. Let M be the midpoint of $\overline{AC}$. E lies on $\overline{BC}$ so that line $\overline{AE}$ is perpendicular to $\overline{BO}$ extended. If $\overline{AC} = 2013$, $\overline{AB} = 2014$, and $\overline{DM} = 249$, find $\overline{CE}$.



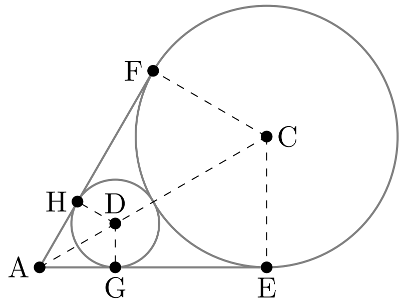
Solution: Notice that because $\overline{AD} + \overline{DC} = \overline{AC} = 2013$ and $\overline{AD} + 249 = \overline{AM} = \frac{2013}{2}$, it follows that $\overline{AD} = \frac{1515}{2}$ and $\overline{DC} = \frac{2511}{2}$. Let F be the point where the incircle intersects $\overline{AB}$ and let G be the point where the incircle intersects $\overline{BC}$. We know that $\overline{AD} = \overline{AF}$ by Power of a Point. Therefore, $\overline{BF} = 2014 - \frac{1515}{2} = \frac{2513}{2}$. However, we also know that $\overline{BF} = \overline{BG}$, and that $\overline{CG} = \overline{CD}$, so it follows that $\overline{BC} = \frac{2513}{2} + \frac{2511}{2} = 2512$. Now notice by symmetry that $\overline{AB} = \overline{BE}$ as $\overline{BO}$ bisects $\angle ABE$. It follows that $\overline{CE} = 2512 - 2014 = \boxed{498}$ as desired.

5. What is the area of a square whose sides are the same length as the sides of an equilateral triangle with area 4?



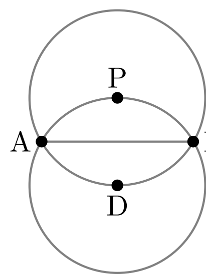
Solution: The area of a square with side length s is s^2 while the area of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$. It follows that the area of the square is $\frac{4}{\sqrt{3}} \cdot 4 = \boxed{\frac{16\sqrt{3}}{3}}$ as desired.

6. Two rays start from a common point and have an angle of 60 degrees. Circle C is drawn with radius 42 such that it is tangent to the two rays. Find the radius of the circle that has radius smaller than circle C and is also tangent to C and the two rays.



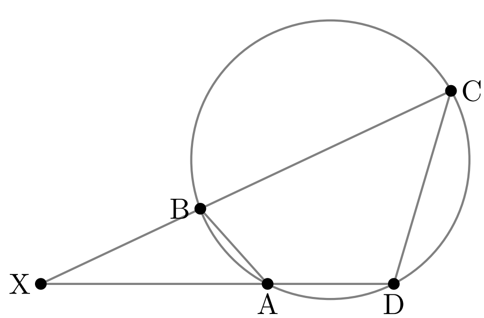
Solution: Label the diagram as shown above. Notice that by symmetry $\angle CAE = 30°$. It follows that $\triangle ACE$ is a $30-60-90$ right triangle, and therefore $\overline{AC} = 84$ and $\overline{AE} = 42\sqrt{3}$. If we let the radius of the smaller circle be r, then similarly we can find that $\overline{AD} = 2r$, and therefore $\overline{AC} = 3r + 42$. It follows that $84 = 42 + 3r$, and therefore $r = \boxed{14}$.

7. Points A and B are fixed points in the plane such that $\overline{AB} = 1$. Find the area of the region consisting of all points P such that $\angle APB > 120°$.



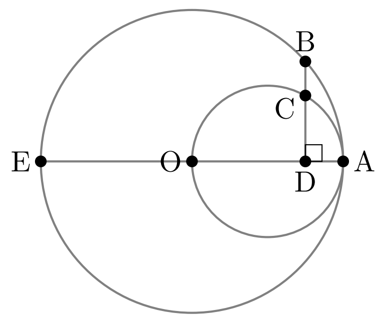
Solution: As shown above, the desired region is the intersection of the two circles which contain points P on arc $\overset{\frown}{AB}$ where $\angle APB = 120°$. Consider the point P shown above such that $\triangle APB$ is isosceles and $\angle APB = 120°$. By symmetry, it follows that $\angle PAB \cong \angle PBA = 30°$. It follows that $\overline{AP} \cong \overline{PB} = \frac{\sqrt{3}}{3}$. Now consider the reflection of P over $\overline{AB}$ and let it be D as shown above. By symmetry, $\overline{PD}$ bisects $\angle APB$, and it follows that $\angle APD \cong \angle DPB \cong \angle PAD \cong \angle PBD = 60°$. It follows that $\triangle APD$ and $\triangle PDB$ are equilateral triangles, and therefore D is the circumcenter of $\triangle APB$. Now it follows that the desired area is double the area of the sector corresponding to arc $\overset{\frown}{A}$ PB of the circle centered at D minus double the area of triangle $\triangle APD$. We can easily determine that this is $\frac{2}{3} \cdot (\frac{\sqrt{3}}{3})^2\pi - \frac{2(\frac{\sqrt{3}}{3})^2\sqrt{3}}{4} = \boxed{\frac{4\pi - 3\sqrt{3}}{18}}$

8. Let $ABCD$ be a cyclic quadrilateral where $\overline{AB} = 4$, $\overline{BC} = 11$, $\overline{CD} = 8$, and $\overline{DA} = 5$. If $\overline{BC}$ and $\overline{DA}$ intersect at X, find the area of $\triangle XAB$.



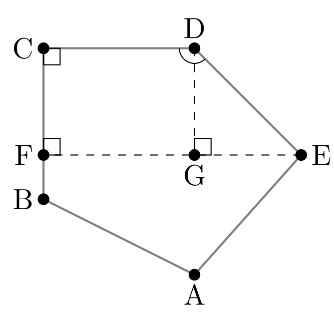
Solution: Notice that because $ABCD$ is cyclic, $\angle CBA = 180° - \angle ADC$, and therefore $\angle XBA \cong \angle XDC$. Similarly, $\angle BAX \cong \angle XCD$. It follows by Angle Angle similarity that $\triangle XBA \sim \triangle XDC$. Let $\overline{XB} = x$ and let $\overline{XA} = y$. It follows that $\frac{x+11}{y} = \frac{y+5}{x} = \frac{8}{4} = 2$. Therefore, $x + 11 = 2y$ and $y + 5 = 2x$. Solving, we get $y = 9$ and $x = 7$. Therefore, by Heron's Formula the area of $\triangle XAB$ is $\sqrt{10 \cdot 1 \cdot 6 \cdot 3} = \boxed{6\sqrt{5}}$.

9. Circle C_1 has center O and radius $\overline{OA}$, and circle C_2 has diameter $\overline{OA}$. $\overline{AB}$ is a chord of circle C_1 and $\overline{BD}$ may be constructed with D on $\overline{OA}$ such that $\overline{BD}$ and $\overline{OA}$ are perpendicular. Let C be the point where C_2 and $\overline{BD}$ intersect. If $\overline{AC} = 1$, find $\overline{AB}$.



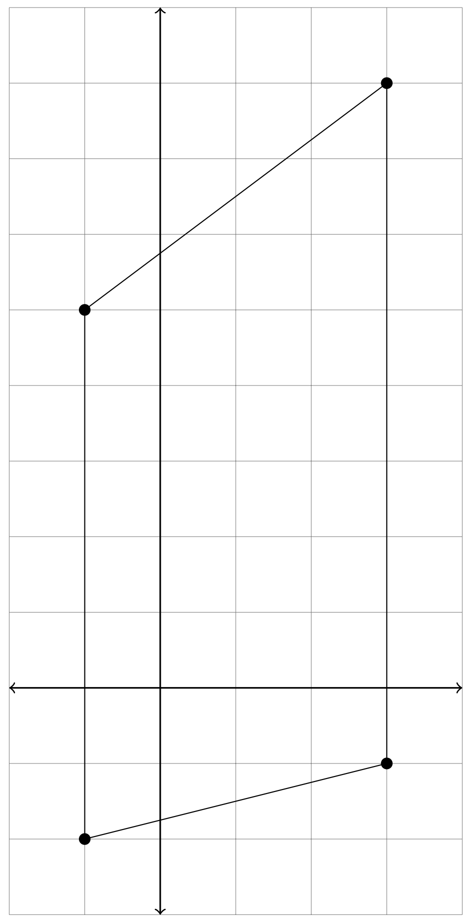
Solution: Let E be the point diametrically opposite of A on C_1 as shown above. Notice that because C lies on the circle with diameter $\overline{OA}$, $\angle OCA$ is a right angle. Similarly, because B lies on the circle with diameter $\overline{AE}$, $\angle EBA$ is a right angle. It follows that $\angle COA = 90° - \angle CAO = \angle ACD$, and $\angle CDA \cong \angle OCA$. It follows that $\triangle CDA \sim \triangle OCA$. Therefore, $\frac{\overline{AC}}{\overline{AD}} = \frac{\overline{OA}}{\overline{AC}}$, and therefore $\overline{AC}^2 = \overline{AD} \cdot \overline{OA}$. Similarly, $\angle BEA = 90° - \angle BAE = \angle DBA$, and because $\angle EBA \cong \angle BDA$, it follows that $\triangle EBA \sim \triangle BDA$. It follows that $\frac{\overline{BA}}{\overline{DA}} = \frac{\overline{EA}}{\overline{BA}}$. It follows that $\overline{BA}^2 = \overline{AD} \cdot \overline{EA}$. It follows that $(\frac{\overline{AB}}{\overline{AC}})^2 = \frac{\overline{EA}}{\overline{OA}} = 2$. It follows that $\overline{AB} = \boxed{\sqrt{2}}$ as desired.

10. Given pentagon $ABCDE$ with $\overline{BC} \cong \overline{CD} \cong \overline{DE} = 4$, $\angle BCD = 90°$, and $\angle CDE = 135°$, what is the length of $\overline{BE}$?



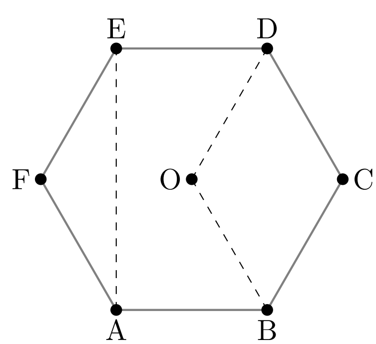
Solution: Label the diagram as shown above. Notice that because $CDGF$ is a rectangle, $\overline{FG} \cong \overline{CD} = 4$. Also, $\angle GDE = 135° - \angle CDG = 45°$. It follows that $\triangle GDE$ is an isosceles right triangle, and therefore $\overline{DG} \cong \overline{GE} = 2\sqrt{2}$. It follows that $\overline{FB} = 4 - 2\sqrt{2}$, and $\overline{FE} = 4 + 2\sqrt{2}$. It follows by the Pythagorean Theorem that $\overline{BE} = \sqrt{(4 + 2\sqrt{2})^2 + (4 - 2\sqrt{2})^2} = \boxed{4\sqrt{3}}$.

11. Find the area of the convex quadrilateral with vertices at the points $(-1, 5), (3, 8), (3, -1)$, and $(-1, -2)$.



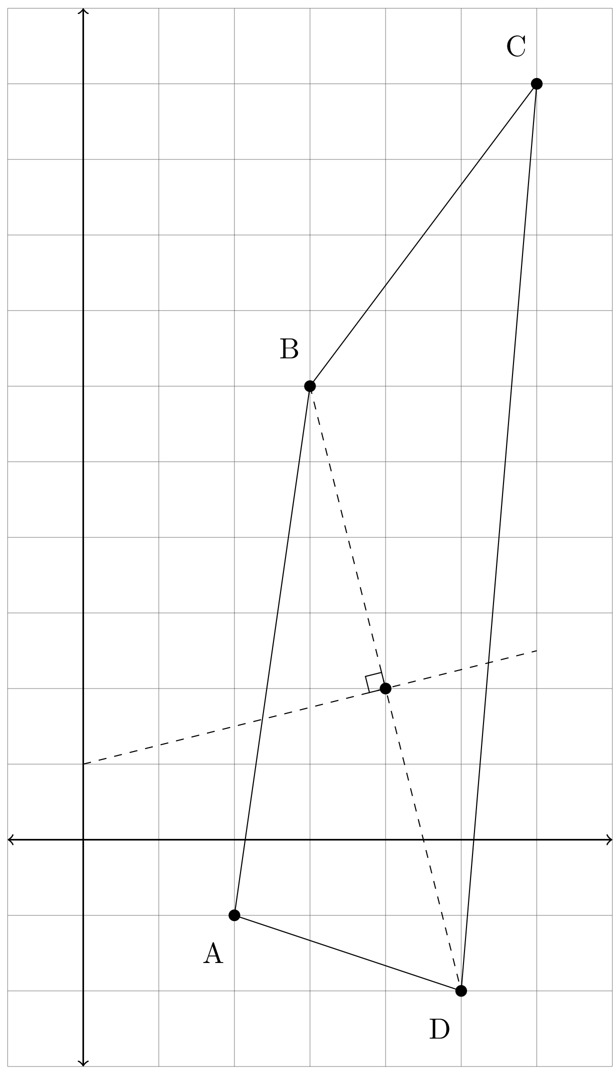
Solution: With a diagram, we can notice that the given quadrilateral is a trapezoid with bases of 9 and 7 and a height of 4. It follows that the area of the quadrilateral is $\frac{(9+7)\cdot 4}{2} = \boxed{32}$. We could also use the Shoelace Formula, but we would need to be careful and make sure that the points are given in clockwise or counterclockwise order.

12. Given regular hexagon $ABCDEF$ with center O and side length 6, what is the area of pentagon $ABODE$?



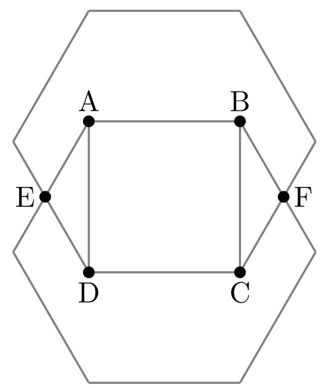
Solution: Notice that the area of $ABODE$ is equivalent to the area when $\triangle ODB$ is removed from the rectangle $EDBA$. Because $\triangle OAB$ and $\triangle OED$ are equilateral triangles with side length 6 by symmetry, $\overline{EA}$ is double the altitude of $\triangle OAB$. By $30-60-90$ right triangles, we can find that the altitude of $\triangle OAB$ is $3\sqrt{3}$, and therefore $\overline{EA} = 6\sqrt{3}$. It follows that the area of $EDBA$ is $6 \cdot 6\sqrt{3} = 36\sqrt{3}$. Now we must subtract the area of $\triangle ODB$. By symmetry, the altitude from O to $\overline{DB}$ is the same as the altitude from O to $\overline{EA}$. It follows that the altitude from O to $\overline{DB}$ is $\frac{AB}{2} = 3$. Therefore, the area of $\triangle ODB$ is $\frac{3 \cdot 6\sqrt{3}}{2} = 9\sqrt{3}$. It follows that the area of $ABODE$ is $36\sqrt{3} - 9\sqrt{3} = \boxed{27\sqrt{3}}$.

13. A quadrilateral $ABCD$ is defined by the points $A = (2, -1)$, $B = (3, 6)$, $C = (6, 10)$, and $D = (5, -2)$. Let l be the line that intersects and is perpendicular to the shorter diagonal at its midpoint. What is the slope of l?



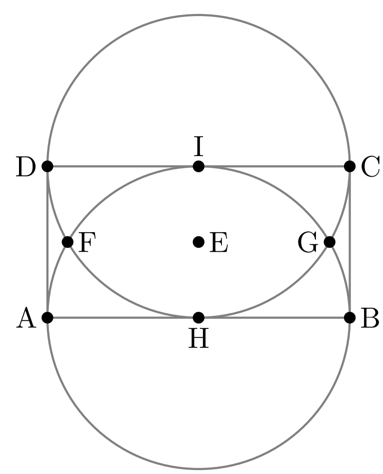
Solution: By the distance formula, $\overline{AC} = \sqrt{137}$ and $\overline{BD} = 2\sqrt{17}$. Therefore, $\overline{BD}$ is the shorter diagonal, and the slope of l is the negative reciprocal of the slope of line $\overline{BD}$. The slope of line $\overline{BD}$ is $\frac{-2-6}{5-3} = -4$, so the slope of l is $\boxed{\frac{1}{4}}$.

14. Given square $ABCD$ with side length 3, we construct two regular hexagons on side $\overline{AB}$ and $\overline{CD}$ such that the hexagons contain the square. What is the area of the intersection of the two hexagons?



Solution: Let the intersection points of the two hexagons be E and F as shown above. Notice that because $\overline{AE}$ is parallel to sides of the hexagon and $\overline{ED}$ lies along an adjacent side of the hexagon, $\angle AED = 120°$. By symmetry, $\angle BFC = 120°$. By symmetry, $\overline{AE} \cong \overline{ED}$. It follows that $\angle EAD \cong \angle EDA = 30°$. It follows by the properties of $30-60-90$ right triangles that the altitude from E to $\overline{AD}$ is $\frac{\sqrt{3}}{2}$. It follows that the area of $\triangle AED$ is $\frac{3\sqrt{3}}{4}$. By symmetry, the area of $\triangle BFC$ is also $\frac{3\sqrt{3}}{4}$. It follows that the area of the intersection of the two hexagons is $3^2+2\cdot\frac{3\sqrt{3}}{4} = \boxed{\dfrac{18+3\sqrt{3}}{2}}$.

15. Let E be a random point inside rectangle $ABCD$ with side lengths $\overline{AB} = 2$ and $\overline{BC} = 1$. What is the probability that angles $\angle AEB$ and $\angle CED$ are both obtuse?



Note: Originally this problem asked for the probability that angles $\angle ABE$ and $\angle CDE$ are obtuse. However, because the probability of this event is obviously 0, I believe the problem writers actually meant for the question to be as stated above. This rewritten problem coincides with the answer on their answer key.

Solution: As shown in the diagram above, the probability that $\angle AEB$ and $\angle CED$ are both obtuse is the ratio of the area of the intersection of the two circles with diameters $\overline{CD}$ and $\overline{AB}$ and the area of $ABCD$. Let I be the midpoint of $\overline{DC}$, let H be the midpoint of $\overline{AB}$, and let the intersections of the two circles be F and G. We know $\overline{FH} \cong \overline{GH} \cong \overline{IH} = \frac{\overline{AB}}{2} = 1$. By symmetry, we also know that $\overline{IF} \cong \overline{IG} = 1$. It follows that $\triangle IFH$ and $\triangle IGH$ are equilateral triangles. It follows taht the area of the intersection of these two circles is made up of 4 $60°$ sectors of a circle of radius 1 minus 2 equilateral triangles of side length 1. It follows that the area of the intersection of these two circles is $\frac{2}{3} \cdot 1^2 \cdot \pi - 2 \cdot \frac{\sqrt{3}}{4} = \frac{4\pi - 3\sqrt{3}}{6}$. It follows that the desired probability is half of this area, or $\boxed{\dfrac{4\pi - 3\sqrt{3}}{12}}$.

3 Sources

1. 2013 Berkeley Math Tournament Spring Individual Problem 2
2. 2013 Berkeley Math Tournament Spring Individual Problem 4
3. 2013 Berkeley Math Tournament Spring Individual Problem 12
4. 2013 Berkeley Math Tournament Spring Individual Problem 14
5. 2014 Berkeley Math Tournament Fall Speed Problem 32
6. 2013 Berkeley Math Tournament Spring Geometry Problem 2
7. 2013 Berkeley Math Tournament Spring Geometry Problem 5
8. 2013 Berkeley Math Tournament Spring Geometry Problem 6
9. 2013 Berkeley Math Tournament Spring Team Problem 5
10. 2014 Berkeley Math Tournament Fall Individual Problem 7
11. 2014 Berkeley Math Tournament Fall Individual Problem 9
12. 2014 Berkeley Math Tournament Fall Individual Problem 14
13. 2014 Berkeley Math Tournament Fall Individual Problem 18
14. 2014 Berkeley Math Tournament Fall Team Problem 7
15. 2014 Berkeley Math Tournament Fall Team Problem 14