Geometry Handout #5 Answers and Solutions

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1 Answers

1. $\sqrt{3}$

2. $\frac{8\sqrt{2}}{9}$

3. 21

4. $\frac{3\sqrt{3}}{2}$

5. $\sqrt{2}$

6. 1

7. $\frac{3+3\sqrt{5}}{2}$

8. 18π

9. $\frac{T^2\pi}{C}$

10. $\frac{348\sqrt{11}}{85}$

11. $2\sqrt{77}$

12. 7

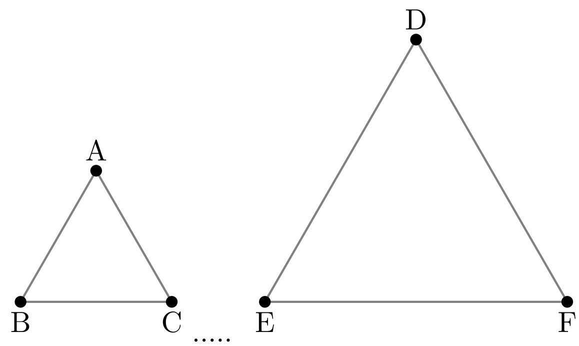
13. 5

14. 24

15. $\frac{2\sqrt{3}}{3}$

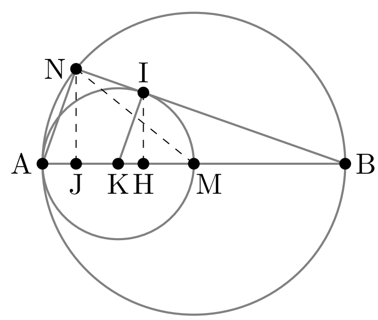
2 Solutions

1. Suppose $\triangle ABC$ is similar to $\triangle DEF$, with A, B, and C corresponding to D, E, and F respectively. If $\overline{AB} = \overline{EF}$, $\overline{BC} = \overline{FD}$, and $\overline{CA} = \overline{DE} = 2$, determine the area of $\triangle ABC$.



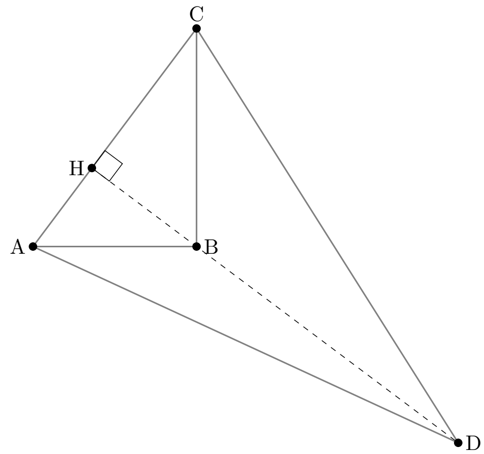
Solution: By the given similarity, we know that $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}}$, or $\frac{\overline{AB}}{2} = \frac{\overline{BC}}{\overline{AB}}$. It follows that $\overline{AB}^2 = 2\overline{BC}$. In addition, we know that $\frac{\overline{AC}}{\overline{DF}} = \frac{\overline{BC}}{\overline{EF}}$ or $\frac{2}{\overline{BC}} = \frac{\overline{BC}}{\overline{AB}}$. It follows that $\overline{BC}^2 = 2\overline{AB}$. It follows that $\overline{BC}^2 = 2\sqrt{2\overline{BC}}$, or $\overline{BC}^4 = 8\overline{BC}$. Because $\overline{BC} \neq 0$, it follows that $\overline{BC} = 2$, from which we can find that $\overline{AB} = 2$, and therefore $\triangle ABC$ is equilateral. It follows that the area of $\triangle ABC$ is $\frac{(2)^2\sqrt{3}}{4} = \boxed{\sqrt{3}}$.

2. Line segment $\overline{AB}$ has length 4 and midpoint M. Let circle C_1 have diameter $\overline{AB}$, and let circle C_2 have diameter $\overline{AM}$. Suppose a tangent of circle C_2 goes through point B to intersect circle C_1 at N. Determine the area of triangle $\triangle AMN$.



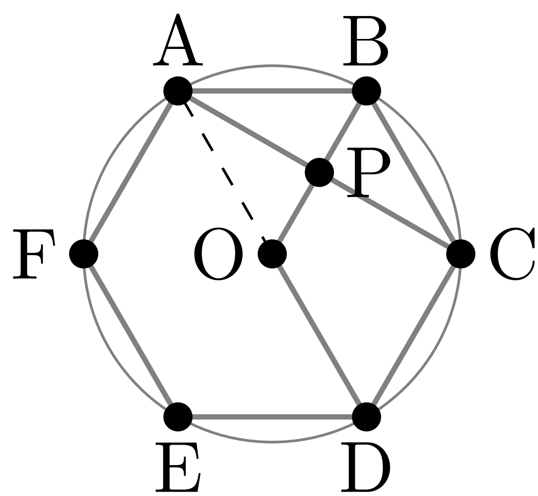
Solution: Let the tangency point between C_2 and $\overline{BN}$ be I, and let the center of C_2 be K as shown in the diagram. We know because $\overline{AB}$ is a diameter of C_1 that $\angle ANB = 90°$. We also know that $\angle KIB = 90°$ because $\overline{NB}$ is tangent to C_2. Finally, we know that $\angle IBK \cong \angle NBA$. It follows that $\triangle KIB$ is similar to $\triangle ANB$. We know that $\frac{\overline{KB}}{\overline{AB}} = \frac{3}{4}$. If we let H be the foot of the perpendicular from I to $\overline{KB}$ and if we let J be the foot of the perpendicular from N to $\overline{AB}$, then it follows that $\frac{\overline{IH}}{\overline{NJ}} = \frac{\overline{KB}}{\overline{AB}} = \frac{3}{4}$. By the Pythagorean Theorem, we can find that $\overline{IB} = 2\sqrt{2}$, and it follows that $\overline{IH} = \frac{1\cdot 2\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$. It follows that $\overline{NJ} = \frac{4}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{8\sqrt{2}}{9}$. Therefore, the area of $\triangle ANM$ is $\frac{8\sqrt{2}}{9} \cdot 2 \cdot \frac{1}{2} = \boxed{\frac{8\sqrt{2}}{9}}$.

3. Suppose four coplanar points A, B, C, and D satisfy $\overline{AB} = 3$, $\overline{BC} = 4$, $\overline{CA} = 5$, and $\overline{BD} = 6$. Determine the maximal possible area of $\triangle ACD$.



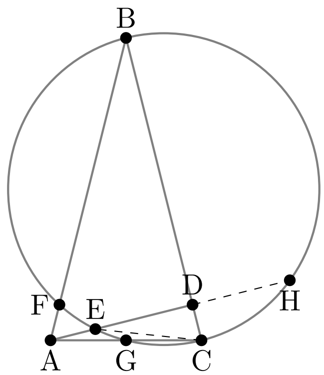
Solution: Clearly the area of $\triangle ACD$ will be maximized when D is as far as possible from $\overline{AC}$. It follows that if H is the foot of the altitude from B to $\overline{AC}$, then we want H, B, and D to be collinear. We know that $\overline{BH} = \frac{3 \cdot 4}{5} = \frac{12}{5}$ by computing the area of $\triangle ABC$ in two different ways. It follows that $\overline{DH} = 6 + \frac{12}{5} = \frac{42}{5}$. Therefore the maximal possible area of $\triangle ACD$ is $\frac{\frac{42}{5} \cdot 5}{2} = \boxed{21}$.

4. Regular hexagon $ABCDEF$ has side length 2 and center O. The point P is defined as the intersection of $\overline{AC}$ and $\overline{OB}$. Find the area of quadrilateral $OPCD$.



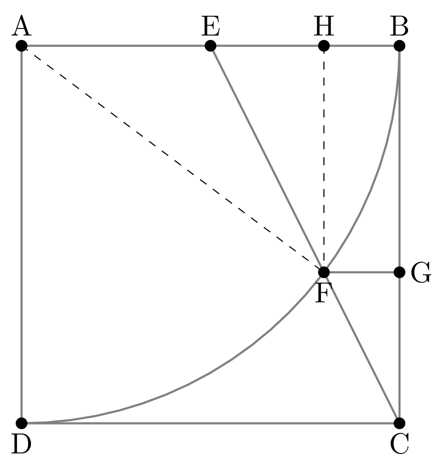
Solution: Notice that by symmetry across the perpendicular bisector of $\overline{FE}$, if D is reflected across O, it will land on A. Also by symmetry across $\overline{BO}$, we know that $\overline{AP} = \overline{PC}$ and that $\angle APO \cong \angle OPC = 90°$. Finally, we know that $\angle ACD = 90°$ as arc $\overset{\frown}{DA}$ is $\frac{3}{4}$ of major arc $\overset{\frown}{(\ \ DB)}$, and therefore $\angle ACD$ is $\frac{3}{4}$ of $\angle BCD$, or $\angle ACD = \frac{3}{4} \cdot 120 = 90°$. In addition, we know that $\angle OAP \cong \angle DAC$, and it follows that $\triangle OAP$ is similar to $\triangle DAC$. It follows by the fact that $\frac{\overline{OA}}{\overline{AD}}$ that the area of $\triangle OAP$ is $\frac{1}{4}$ of the area of $\triangle DAC$, or that the area of $OPCD$ is $\frac{3}{4}$ of the area of $\triangle ADC$. $\overline{DC}$ has a length of 2. Because $\angle BAC \cong \angle BCA = \frac{180-120}{2} = 30°$, we know that $\overline{AP} = \sqrt{3}$ due to the nature of $30-60-90$ triangles. It follows that the area of $\triangle ACD$ is $\frac{2 \cdot 2\sqrt{3}}{2} = 2\sqrt{3}$, and from here we can find that the area of $OPCD$ is $\frac{3}{4} \cdot 2\sqrt{3} = \boxed{\dfrac{3\sqrt{3}}{2}}$.

5. Consider an isosceles triangle $\triangle ABC$ ($\overline{AB} = \overline{BC}$). Let D be on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$ and O be a circle with diameter $\overline{BC}$. Suppose that segment $\overline{AD}$ intersects circle O at E. If $\overline{CA} = 2$, what is $\overline{CE}$?



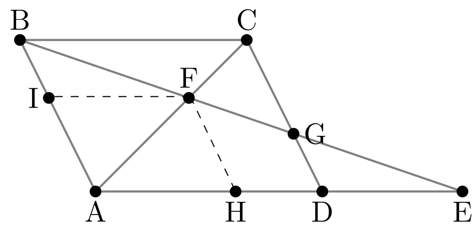
Solution: Let F be the other intersection of O with $\overline{AB}$ aside from B and let G be the other intersection of O with $\overline{AC}$ aside from C. Because $\overline{BC}$ is a diameter of O, we know that $\angle BGC = 90°$, and because $\overline{AB} = \overline{BC}$, we know that $\overline{AG} = \overline{GC} = 1$. Similarly, we know that $\angle BFC = 90°$, and it follows by symmetry that $\overline{AF} = \overline{DC}$ and that $\overline{BF} = \overline{BD}$. Let $\overline{AF} = \overline{DC} = x$, and let $\overline{BF} = \overline{BD} = y$. Let ED intersect O again at H. By symmetry, we know that $\overline{ED} = \overline{DH}$. By Power of a Point on D, we know that $xy = \overline{ED}^2$. It follows by the Pythagorean Theorem on $\triangle EDC$ that $\overline{EC}^2 = x^2 + xy$. By Power of a Point on A, we know that $x \cdot (x + y) = 1 \cdot (1 + 1) = 2$. It follows that $\overline{EC}^2 = x \cdot (x + y) = 2$, and this gives us an answer of $\overline{EC} = \boxed{\sqrt{2}}$.

6. Square $ABCD$ has side length 5 and arc $\overset{\frown}{BD}$ with center A. E is the midpoint of $\overline{AB}$ and $\overline{CE}$ intersects arc $\overset{\frown}{BD}$ at F. G is placed onto $\overline{BC}$ such that $\overline{FG}$ is perpendicular to $\overline{BC}$. What is the length of $\overline{FG}$?



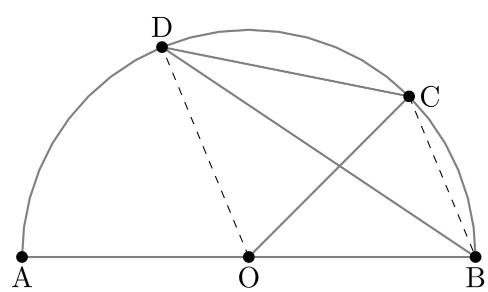
Solution: Let H be the foot of the perpendicular from F to $\overline{AB}$. Let $\overline{FG} = x$. Because $\triangle FCG$ is similar to $\triangle ECB$, and $\frac{\overline{EB}}{\overline{BC}} = \frac{1}{2}$, we know that $\overline{GC} = 2x$. It follows because $HBGF$ is a rectangle that $\overline{HF} = 5 - 2x$ and that $\overline{AH} = 5 - x$. Using the Pythagorean Theorem on $\triangle AFH$, we get $25 = (5-x)^2 + (5-2x)^2$. This rearranges to $5x^2 - 30x + 25 = 0$. Solving this quadratic, we get $x = 1$ or $x = 5$. Because $\overline{FG}$ is clearly less than 5, our answer is $\overline{FG} = \boxed{1}$.

7. Consider a parallelogram $ABCD$. E is a point on ray $\vec{AD}$. $\overline{BE}$ intersects $\overline{AC}$ at F and $\overline{CD}$ at G. If $\overline{BF} = \overline{EG}$ and $\overline{BC} = 3$, find the length of $\overline{AE}$.



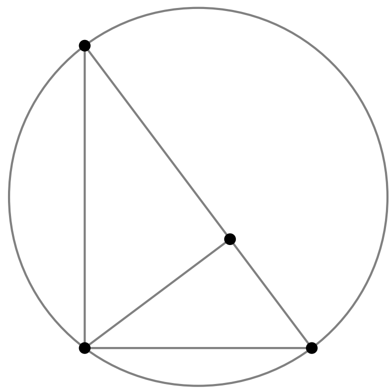
Solution: Let the intersection of $\overline{BE}$ and $\overline{AC}$ be F and let the intersection of $\overline{CD}$ and $\overline{BE}$ be G as shown above. Let H lie on $\overline{AE}$ such that $\overline{FH}$ is parallel to $\overline{AB}$, and let I lie on $\overline{AB}$ such that $\overline{FI}$ is parallel to $\overline{AE}$. Let $\overline{GE} = \overline{BF} = x$, and let $\overline{DE} = ax$ for some constant a. Notice that because $\triangle BIF$ and $\triangle GDE$ are both clearly similar to $\triangle ABE$, we must have that $\triangle BIF$ is similar to $\triangle GDE$. In addition, because $\overline{BF} = \overline{EG} = x$, we must have that $\triangle BIF$ is congruent to $\triangle GDE$, and therefore $\overline{IF} = \overline{DE} = ax$. Because $ABFI$ is a parallelogram, we must have $\overline{AB} = ax$. It follows that $\overline{HD} = 3 - ax$, and because $\triangle GDE$ is similar to $\triangle HEF$, we must have $\frac{\overline{DE}}{\overline{GE}} = \frac{\overline{HE}}{\overline{FE}} = a$, and it follows that $\overline{FG} = \frac{3-ax}{a}$ and $\overline{FE} = x + \frac{3-ax}{a} = \frac{3}{a}$. Because $\triangle BFC$ is similar to $\triangle EFA$, we know that $\frac{\overline{FE}}{\overline{BF}} = \frac{\overline{AE}}{\overline{BC}} = \frac{3+ax}{3}$. Therefore, $\frac{\frac{3}{a}}{x} = \frac{3+ax}{3} = 1 + \frac{ax}{3}$. Letting $ax = y$, we get $\frac{3}{y} = 1 + \frac{y}{3}$ which rearranges to $y^2 + 3y - 9 = 0$, and solving we get $y = \frac{3\sqrt{5}-3}{2}$. It follows that $\overline{AE} = 3 + ax = 3 + y = \boxed{\dfrac{3+3\sqrt{5}}{2}}$ as desired.

8. Semicircle O has diameter $\overline{AB} = 12$. Arc $\overset{\frown}{AC} = 135°$. Let D be the midpoint of arc $\overset{\frown}{AC}$. Compute the (area of the) region bounded by the lines $\overline{CD}$ and $\overline{DB}$ and the arc $\overset{\frown}{CB}$.



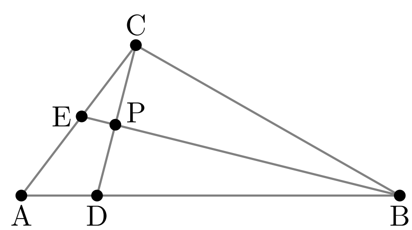
Solution: Notice that the area we want is equal to the sum of the areas of $\triangle OCB$ and sector $\angle COB$ of semicircle O minus the area of $\triangle ODB$. The area of sector $\angle COB$ of semicircle O is clearly $(12)^2\pi \cdot \frac{45°}{360°} = 18\pi$. The area of $\triangle OCD$ is $12 \cdot 12 \cdot \sin(\frac{135°}{2}) \cdot \frac{1}{2}$. The area of $\triangle ODB$ is $12 \cdot 12 \cdot \sin(\frac{225°}{2}) \cdot \frac{1}{2}$. Because $\sin(\frac{135^{circ}}{2}) = \sin(180° - \frac{135^{circ}}{2}) = \sin(\frac{225^{circ}}{2})$, it follows that the area of $\triangle OCD$ is equivalent to the area of $\triangle ODB$, and therefore the area we want is just the area of sector $\angle COB$ which is clearly just $\boxed{18\pi}$.

9. In a right triangle, the altitude from a vertex to the hypotenuse splits the hypotenuse into two segments of lengths a and b. If the right triangle has area T and is inscribed in a circle of area C, find ab in terms of T and C.



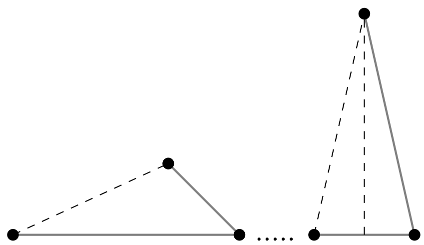
Solution: It is well known that the circumcenter of a right triangle lies at the midpoint of its hypotenuse. It follows that the circumradius of our right triangle is $\frac{a+b}{2}$, and therefore the area of the circumcircle is $C = (\frac{a+b}{2})^2\pi$. If we let the altitude to the hypotenuse be h, then by similar triangles we can find that $\frac{a}{h} = \frac{h}{b}$, from which we can obtain $h = \sqrt{ab}$. It follows that the area of the right triangle is $T = \frac{(a+b)\sqrt{ab}}{2}$. From here it is clear that $ab = \boxed{\frac{T^2\pi}{C}}$

10. Let $\triangle ABC$ be a triangle with $\overline{AB} = 16$, $\overline{AC} = 10$, $\overline{BC} = 18$. Let D be a point on $\overline{AB}$ such that $4\overline{AD} = \overline{AB}$ and let E be the foot of the angle bisector from B onto $\overline{AC}$. Let P be the intersection of $\overline{CD}$ and $\overline{BE}$. Find the area of the quadrilateral $ADPE$.



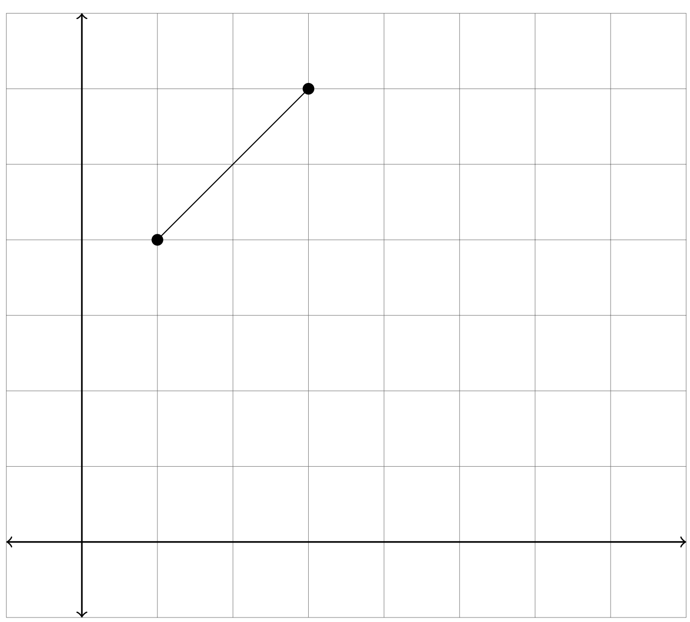
Solution: We will proceed by mass points. By the Angle Bisector Theorem, we know that $\frac{\overline{AE}}{\overline{EC}} = \frac{16}{18} = \frac{8}{9}$. Therefore, we will let the mass at A be 9 and we will let the mass at C be 8. It follows that the mass at E is 17. Because $\frac{\overline{AD}}{\overline{DB}} = \frac{1}{3}$, we will let the mass at B be $\frac{9}{3} = 3$. It follows that the mass at point D is $9 + 3 = 12$, and therefore $\frac{\overline{CP}}{\overline{PD}} = \frac{3}{2}$. With this information, we can express the areas of certain triangles as fractions of the area of $\triangle ABC$. We know that $\triangle ADC$ and $\triangle BCD$ have the same height from C to the opposite side, so it follows that the ratio of their areas is the same as the ratio of their bases, and it follows that the area of $\triangle BCD$ is $\frac{3}{4}$ of the area of $\triangle ABC$. In addition, we know that $\triangle BPC$ and $\triangle BPD$ share the altitude from B to the opposite side, so we know that the ratio of their areas is the same as the ratio of their bases. It follows that the area of $\triangle BPD$ is $\frac{2}{5}$ of the area of $\triangle BCD$. It follows that the area of $\triangle BPD$ is $\frac{3}{4} \cdot \frac{2}{5} = \frac{3}{10}$ of the area of $\triangle ABC$. In addition, we know that triangles $\triangle BEC$ and $\triangle ABE$ share the altitude from B to the opposite side, so the ratio of their areas is the same as the ratio of their bases. It follows that the area of $\triangle AEB$ is $\frac{8}{17}$ of the area of $\triangle ABC$. It follows that the area of $ADPE$ is $\frac{8}{17} - \frac{3}{10} = \frac{29}{170}$ of the area of $\triangle ABC$. By Heron's Formula, the area of $\triangle ABC$ is $\sqrt{22 \cdot 6 \cdot 12 \cdot 4} = 24\sqrt{11}$, and it follows that the area of $ADPE$ is $\frac{29}{170} \cdot 24\sqrt{11} = \boxed{\frac{348\sqrt{11}}{85}}$.

11. Two sides of an isosceles triangle $\triangle ABC$ have lengths 9 and 4. What is the area of $\triangle ABC$?



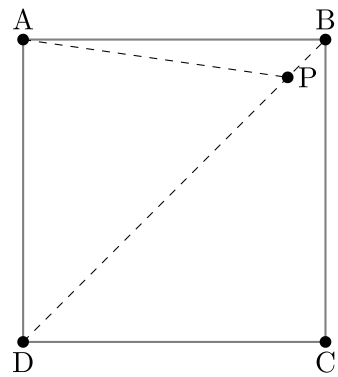
Solution: Notice that if the two equal sides of $\triangle ABC$ were both of length 4, then $\triangle ABC$ would not satisfy the Triangle Inequality as $4 + 4 = 8 < 9$. Therefore, the two equal sides are both of length 9 and they are connected to a base of length 4. It follows that the foot of the perpendicular from the vertex of $\triangle ABC$ which is not on the base is the midpoint of the base and divides it into two parts which are each equal to 2. By the Pythagorean Theorem, it follows that the altitude is $\sqrt{77}$, and therefore the area of $\triangle ABC$ is $\boxed{2\sqrt{77}}$.

12. An isosceles triangle has two vertices at $(1, 4)$ and $(3, 6)$. Find the x-coordinate of the third vertex assuming it lies on the x-axis.



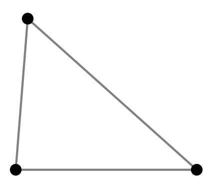
Solution: Notice that it is impossible for the given side to be one of the two equal sides of the isosceles triangle as the distance between these two points is $2\sqrt{2}$ and they are each at least 4 from the x-axis. It follows that the third vertex is equidistant from each of the two given points, and therefore it lies on the perpendicular bisector of the given side. The perpendicular bisector of the given side has the equation $y = 7 - x$, so it follows that the third point has coordinates $(7, 0)$, and therefore its x-coordinate is $\boxed{7}$.

13. A point P is inside the square $ABCD$. IF $\overline{PA} = 5$, $\overline{PB} = 1$, and $\overline{PD} = 7$, then what is $\overline{PC}$?



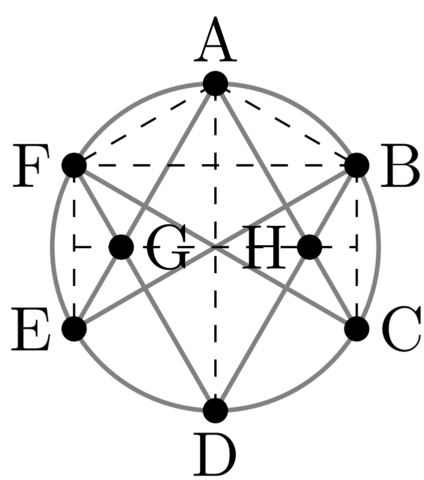
Solution: By the British Flag Theorem, we know that $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$. Plugging in the given lengths, it follows that $\overline{PC}^2 = 7^2 + 1^2 - 5^2 = 25$, and from here it follows that $\overline{PC} = \boxed{5}$.

14. Two sides of a triangle have lengths 20 and 30. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?



Solution: Let the altitude to the side of length 20 be h and let the altitude to the side of length 30 be y. It follows that $20h = 30y$, and therefore $h = \frac{3}{2}y$. It follows that the altitude to the third side is $\frac{y+\frac{3}{2}y}{2} = \frac{5}{4}y$. If we let the third side be s, then it follows that $30y = \frac{5}{4}ys$, from which it follows that $s = \boxed{24}$.

15. Assume the A, B, C, D, E, and F are equally spaced on a circle of radius 1, as in the figure below. Find the area of the kite bounded by the lines $\overline{EA}$, $\overline{AC}$, $\overline{FC}$, and $\overline{BE}$.



Solution: Notice that because $\overset{\frown}{EC}$ is $\frac{1}{2}$ of major arc $\overset{\frown}{FB}$, $\angle EAC \cong \angle FDB$ is equal to $\frac{1}{2}$ of $\angle FAB$. Therefore, $\angle EAC \cong \angle FDB = \frac{1}{2} \cdot 120 = 60°$. Now let the intersection point of $\overline{AE}$ and $\overline{FD}$ be G and let the intersection point of $\overline{AC}$ and $\underline{BD}$ be H as shown above. With the given information, we can easily deduce that $\angle AGD \cong \angle AHD \cong \angle FGE \cong \angle BHC = 120°$. By symmetry, we know that $\triangle FGE$ is isosceles, and it follows that $\angle GFE \cong \angle FEG = 30°$. It follows that the triangle formed by F, G, and the foot of the perpendicular from is a $30 - 60 - 90$ right triangle with a longer leg of length $\frac{1}{2}$. It follows that the distance from G to the foot of the perpendicular from G to $\overline{FE}$ is $\frac{\sqrt{3}}{6}$. Similarly, the distance from H to the foot of the perpendicular from H to $\overline{BC}$ is also $\frac{\sqrt{3}}{6}$. Finally, we can split $\triangle AFB$ into two $30 - 60 - 90$ right triangles each with a hypotenuse of length 1, and it follows that $\overline{FB} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$. It follows that $\overline{GH} = \sqrt{3} - \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$. In addition, by symmetry we know that triangles $\triangle AGH$ and $\triangle GHD$ are equilateral, and therefore the distance from A to D is $2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{2\sqrt{3}}{3} = 2$. It follows that the area of $AGDH$ is $2 \cdot \frac{2\sqrt{3}}{3} \cdot \frac{1}{2} = \boxed{\dfrac{2\sqrt{3}}{3}}$.

3 Sources

1. 2014 Berkeley Math Tournament Spring Individual Problem 2
2. 2014 Berkeley Math Tournament Spring Individual Problem 8
3. 2014 Berkeley Math Tournament Spring Individual Problem 12
4. 2014 Berkeley Math Tournament Spring Geometry Problem 2
5. 2014 Berkeley Math Tournament Spring Geometry Problem 3
6. 2014 Berkeley Math Tournament Spring Geometry Problem 6
7. 2014 Berkeley Math Tournament Spring Geometry Problem 7
8. 2014 Berkeley Math Tournament Spring Geometry Problem 8
9. 2014 Berkeley Math Tournament Spring Team Problem 4
10. 2014 Berkeley Math Tournament Spring Team Problem 13
11. 2015 Berkeley Math Tournament Fall Individual Problem 3
12. 2015 Berkeley Math Tournament Fall Individual Problem 9
13. 2015 Berkeley Math Tournament Fall Individual Problem 13
14. 2015 Berkeley Math Tournament Fall Individual Problem 16
15. 2015 Berkeley Math Tournament Fall Individual Problem 18