Combinatorics Handout 2 Answers and Solutions

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1 Answers

1. $\frac{19}{27}$

2. $\frac{3}{10}$

3. $\frac{6}{7}$

4. 2400

5. $\frac{2012}{2013}$

6. $\frac{4}{5}$

7. 63

8. 120

9. $\frac{1}{3}$

10. 10080

11. $\frac{2048}{4095}$

12. 1394

13. $\frac{363}{20}$

14. 10800

15. $\frac{64}{15}$

2 Solutions

1. If you roll three fair, six-sided dice, what is the probability that the product of the results will be a multiple of 3?

Solution: In order for the product of the results to be a multiple of 3, at least one of the dice must roll a 3 or a 6. The probability that any dice will not roll a 3 or a 6 is $\frac{4}{6} = \frac{2}{3}$. It follows that the probability that none of the dice will roll a 3 or a 6 is $(\frac{2}{3})^3 = \frac{8}{27}$. Therefore, the probability that at least one of the dice will roll a 3 or a 6 is $1 - \dfrac{8}{27} = \boxed{\dfrac{19}{27}}$ as desired.

2. Derek has 10 American coins in his pocket, summing to a total of 53 cents. If he randomly grabs 3 coins from his pocket, what is the probability that theyre all different?

Solution: We can notice that Derek must have either 3 or 8 pennies as all other coins are multiples of 5, and therefore $53 - p$ must be a multiple of 5 where p is the number of pennies. Notice that if

Derek had 8 pennies, he would have to make 45¢ with only 2 coins. This is clearly impossible, so Derek must have 3 pennies. Now Derek must make 50¢ with 7 coins. Either by guess and check, or factoring, we can find that Derek must have 4 nickels and 3 dimes. It follows that the probability that Derek grabs 3 different coins from his pocket is $\dfrac{3 \cdot 4 \cdot 3}{\binom{10}{3}} = \boxed{\dfrac{3}{10}}$ as desired.

3. 15 people, including Luke and Matt, attend a Berkeley Math meeting. If Luke and Matt sit next to each other, a fight will break out. If they sit around a circular table, all positions equally likely, what is the probability that a fight doesnt break out?

Solution: Fix Luke's position on the table. Matt has 14 positions to sit in relative to *Luke*, and only 2 of these will cause a fight to break out, so our answer is $1 - \dfrac{2}{14} = \boxed{\dfrac{6}{7}}$ as desired.

4. How many ways can you arrange the letters of the word BERKELEY such that no two Es are next to each other?

Solution: Notice that any rearrangement of the letters in BERKELEY with this property must be of the form

-E-E-E-

where the dashes between the E's must have letters and the dashes on the sides could have letters. We can fill the dashes with the letters B,R,K,L,Y to get rearrangements of the letters in BERKELEY. By Stars and Bars, The number of ways to divide 5 letters into four sections such that each section has at least one letter is $\binom{4}{3} = 4$. The number of ways to divide 5 letters into three sections such that each section has at least one letter is $\binom{4}{2} = 6$. Finally, the number of ways to divide 5 letters into two sections such that each section has at least one letter is $\binom{4}{1} = 4$. We can separate the letters into three sections in two different ways: one where a section is to the left of the first E and one where a section is to the right of the third E. Therefore, there are $4 + 2 \cdot 6 + 4 = 20$ ways to divide the letters. However, we must multiply by 5! to rearrange these letters, so our answer is $20 \cdot 120 = \boxed{2400}$ as desired.

5. Tom has 2012 blue cards, 2012 red cards, and 2012 boxes. He distributes the cards in such a way such that each box has at least 1 card. Sam chooses a box randomly, then chooses a card randomly. Suppose that Tom arranges the cards such that the probability of Sam choosing a blue card is maximized. What is this maximum probability?

Solution: Clearly, to maximize the chances of picking a blue card, we want as many boxes with only blue cards as possible. This can be done by placing 1 blue card in each of 2011 boxes, and placing the remaining blue card and all of the red cards in the 2012th box. This gives a probability of picking a blue card of

$$\frac{2011}{2012} + \frac{1}{2012} \cdot \frac{1}{2013} = \frac{2012^2}{2012 \cdot 2013} = \boxed{\frac{2012}{2013}}$$

6. A bag holds 6 coins. Three have tails on both sides, two have heads on both sides, and one has heads on one side and tails on the other. If you pick a coin at random and notice the only side you can see is heads, what is the probability that the other side is also a head?

Solution: If we see a head on one side of a coin, obviously the coin cannot have tails on both sides. Therefore, the only possible coins are the two coins with heads on both sides and the one normal coin. The probability of selecting a coin with heads on both sides and seeing a head is $\frac{2}{3} \cdot 1 = \frac{2}{3}$. The probability of selecting the normal coin and seeing a head is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. Therefore our answer is

$$\frac{\frac{2}{3}}{\frac{2}{3} + \frac{1}{6}} = \boxed{\frac{4}{5}}$$

7. How many solutions (x, y) in the positive integers are there to $3x + 7y = 1337$?

Solution: Notice that the solution (x, y) with the largest value of y is $(7, 188)$. The only way to generate other solutions is to decrease the y-value by 3 and increase the x-value by 7. Therefore, our answer is $\lceil \frac{188}{3} \rceil = \boxed{63}$ as desired.

8. Brandon is located at $(3, 2)$, and Chuck is located at $(6, -5)$. Brandon can only move unit distance to the right or down, and Chuck is stationary. In how many different ways can Brandon move to Chuck?

Solution: Notice that Brandon must move 3 units to the right and 7 units down to get to Chuck. It follows that we can form a one to one correspondence between paths to Chuck and rearrangements of the letters RRRDDDDDDD. The number of ways to rearrange these letters is

$$\frac{10!}{3! \cdot 7!} = \boxed{120}$$

9. Four points A, B, C, and D are randomly chosen on a circle. What is the probability that AB and CD intersect inside the circle?

Solution: Fix the position of point A on the circle. Notice that if going clockwise from A, B is encountered after another point not equal to A is encountered, then AB and CD must intersect inside the circle. The probability that B is between points C and D when doing this is $\boxed{\frac{1}{3}}$ as desired.

10. How many ways can you arrange the letters in MATHISHARD such that the permutation begins with MR?

Solution: Fix the letters MR at the beginning of the permutation. It follows that the rest of the permutation is a rearrangement of the letters in ATHISHAD. The number of rearrangements of these letters is

$$\frac{8!}{2! \cdot 2!} = \boxed{10080}$$

11. There are 12 people labeled $1, 2, \cdots 12$ working together on 12 missions, with persons $1, 2, \cdots i$ working on the ith mission. There is exactly one spy among them. If the spy is not working on a mission, it will be a huge success, but if the spy is working on the mission, it will fail with probability $\frac{1}{2}$. Given that the first 11 missions succeed, and the 12th mission fails, what is the probability that person 12 is the spy?

Solution: Notice that given person i is the spy, the probability that the first 11 missions will succeed and the 12th mission will fail is $(\frac{1}{2})^{13-i}$. It follows that our probability is

$$\frac{\frac{1}{2}}{\sum_{x=1}^{12} (\frac{1}{2})^{13-x}} = \boxed{\frac{2048}{4095}}$$

12. In prokaryotes, translation of mRNA messages into proteins is most often initiated at start codons on the mRNA having the sequence AUG. Assume that the mRNA is single-stranded and consists of a sequence of bases, each described by a single letter, A,C,U, or G. Consider the set of all pieces of bacterial mRNA six bases in length. How many such mRNA sequences have either no As or no Us?

Solution: In order for a mRNA sequence to not have any A's, all of the bases must be C's, U's or G's. The number of sequences with this property is $3^6 = 729$. Similarly, the number of sequences with no U's is $3^6 = 729$. However, simply adding these two would double count the cases where there are both no A's and no U's. The number of sequences with this property is $2^6 = 64$. Therefore our answer is $2 \cdot 729 - 64 = \boxed{1394}$ as desired.

13. Inside a LilacBall, you can find one of 7 different notes, each equally likely. Delcatty must collect all 7 notes in order to restore harmony and save Kanto from eternal darkness. What is the expected number of LilacBalls she must open in order to do so?

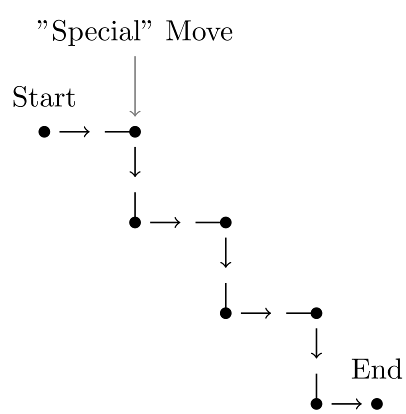
Solution: Let E_n be the expected number of LilacBalls Delcatty must open to collect the 7 notes given she has already found n notes. $E_7 = 0$ and we wish to find E_0. Notice that we can get the recurrence $E_n = \frac{n}{7} \cdot E_n + \frac{7-n}{7} \cdot E_{n+1} + 1 \rightarrow E_n = E_{n+1} + \frac{7}{7-n}$. Therefore our answer is

$$\sum_{x=0}^{6} \frac{7}{7-n} = \boxed{\frac{363}{20}}$$

Note: In the official answer key for this test, an answer of $\frac{657}{80}$ was given. The official solution gives the same expression as the one in the solution above for the answer. A check with Wolfram Alpha reveals that the answer above is correct.

14. Link starts at the top left corner of an 12×12 grid and wants to reach the bottom right corner. He can only move down or right. A turn is defined a down move immediately followed by a right move, or a right move immediately followed by a down move. Given that he makes exactly 6 turns, in how many ways can he reach his destination?

Solution: Clearly Link must do 3 turns which are downs followed by rights and 3 turns which are rights followed by downs. We can think of turns which are down followed by rights as making a "special" move downwards, and then immediately afterwards there is either a normal right move or a "special" right move which immediately becomes a turn. Therefore, we need 3 "special" right moves and 3 "special" down moves in addition to 8 normal right moves and 8 normal down moves. There are two possibilities. Either Link's first turn is a right followed by a down or it is a down followed by a right. The diagram below shows what the first possibility might look like:



In the first possibility, Link must divide his 7 remaining right moves among 4 different sections of his trip after adding a right move to the final turn, and he must divide his 8 remaining down moves among 3 different sections of his trip. By Stars and Bars, the number of ways to do this is $\binom{10}{3} \cdot \binom{10}{2} = 5400$. By symmetry, there will be the same number of ways in the second possibility so our answer is $2 \cdot 5400 = \boxed{10800}$ as desired.

15. Katniss has an n-sided fair die which she rolls. If $n > 2$, she can either choose to let the value rolled be her score, or she can choose to roll a $n-1$ sided fair die, continuing the process. What is the expected value of her score assuming Katniss starts with a 6-sided die and plays to maximize this expected value.

Solution: Let E_n be the expected value when beginning with a n-sided die. Clearly, $E_1 = 1$, and $E_2 = \frac{1+2}{2} = \frac{3}{2}$. Notice that Katniss will not bother rolling an $n-1$-sided fair die if she rolls anything higher than E_{n-1}. It follows that $E_3 = \frac{1}{3} \cdot E_2 + \frac{2+3}{3} = \frac{13}{6}$. Continuing this process, we get $E_4 = \frac{2}{4} \cdot E_3 + \frac{3+4}{4} = \frac{17}{6}$, $E_5 = \frac{2}{5} \cdot E_4 + \frac{3+4+5}{5} = \frac{53}{15}$, and finally $E_6 = \frac{3}{6} \cdot E_5 + \frac{4+5+6}{6} = \boxed{\frac{64}{15}}$ as desired.

3 Sources

1. Berkeley Math Tournament Individual Fall 2012 Problem 3
2. Berkeley Math Tournament Individual Fall 2012 Problem 10
3. Berkeley Math Tournament Individual Fall 2012 Problem 13
4. Berkeley Math Tournament Individual Fall 2012 Problem 15
5. Berkeley Math Tournament Individual Fall 2012 Problem 17
6. Berkeley Math Tournament Gambling Fall 2012 Problem 1
7. Berkeley Math Tournament Team Fall 2012 Problem 4
8. Berkeley Math Tournament Countdown Fall 2012 Problem 4
9. Berkeley Math Tournament Countdown Fall 2012 Problem 5
10. Berkeley Math Tournament Countdown Fall 2012 Problem 8
11. Berkeley Math Tournament Team Spring 2012 Problem 4
12. Berkeley Math Tournament Tournament Round 1 Spring 2012 Problem 3
13. Berkeley Math Tournament Tournament Round 2 Spring 2012 Problem 6
14. Berkeley Math Tournament Tournament Round 4 Spring 2012 Problem 6
15. Berkeley Math Tournament Championship Round Spring 2012 Problem 2