Combinatorics Handout 3 Answers and Solutions

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1 Answers

1. $\frac{1}{16}$

2. 10

3. $MTMB$

4. 225

5. 4

6. 1858

7. 34

8. 89

9. 320

10. 11

11. 35

12. 600

13. $\frac{1}{120}$

14. $\frac{143}{4096}$

15. $\frac{25}{64}$

2 Solutions

1. You toss five coins one after another. What is the probability that you never get two consecutive heads or two consecutive tails?

Solution: In order to never have two consecutive coin flips with the same side, the five coins must alternate in their sides. If we let H represent a head and T represent tails, then we have that the only possibilities for the five coins are $HTHTH$ and $THTHT$. The total number of possibilities is $2^5 = 32$. Therefore our answer is $\frac{2}{32} = \boxed{\frac{1}{16}}$.

2. Out of 100 customers at a market, 80 purchased oranges, 60 purchased apples, and 70 purchased bananas. What is the least possible number of customers who bought all three items?

Solution: Notice that there were $100-80=20$ people who did not purchase oranges, $100-60=40$ people who did not purchase apples, and $100-70=30$ people who did not purchase bananas. To minimize the number of people who bought all three islands, we want the people from each of these groups to all be distinct, so as to maximize the number of people who did not buy one of the items. Therefore, our answer is $100-20-30-40=\boxed{10}$.

3. Twelve distinct four-letter words can be formed by using each of the letters B, M, M, and T exactly once. What is the ninth word when these words are arranged in alphabetical order?

Solution: The number of words that begin with B is the number of rearrangements of MMT, which is $\frac{3!}{2!}=3$. These words will come first in alphabetical order. The number of words that begin with M is the number of rearrangements of BMT which is $3!=6$. Because $3+6=9$, we want the last of these rearrangements when written in alphabetical order. Therefore, our answer is $\boxed{MTMB}$

4. How many three digit even numbers are there with an even number of even digits?

Solution: If a three digit number is even, its units digit must be even. If it has an even number of even digits, then exactly one of the other two digits must be even. We will do casework on which digit it is:

Case 1: The hundreds digit is even.

Because the hundreds digit cannot be 0, we have 4 possibilities $(2,4,6,8)$ for the hundreds digit, 5 possibilities for the tens digit (as it must be odd), and 5 possibilities for the units digit. Therefore, we have a total of $4\cdot 5\cdot 5=100$ possibilities in this case.

Case 2: The tens digit is even.

In this case, we have 5 possibilities for the hundreds digit, 5 possibilities for the tens digit, and 5 possibilities for the units digit, for a total of $5\cdot 5\cdot 5=125$ possibilities in this case.

Adding up our cases, we have a total of $\boxed{225}$ numbers which satisfy these properties.

5. Three boys, Bob, Charles, and Derek, and three girls, Alice, Elizabeth, and Felicia are all standing in one line. Bob and Derek are each adjacent to precisely one girl, while Felicia is next to two boys. If Alice stands before Charles, who stands before Elizabeth, determine the number of possible ways they can stand in a line.

Solution: Let B represent Bob, C represent Charles, D represent Derek, A represent Alice, E represent Elizabeth, and F represent Felicia. By the third condition, we know they are ordered in some permutation of the form $?A?C?E?$ where each of the question marks can have any number of people. Notice because B and D are next to precisely one girl, they are either on the ends, or they are adjacent to one other boy. Notice if B were adjacent to C, F would need to be adjacent to D and one of B or C. Either way, D would be adjacent to two girls, which is impossible. Therefore, neither B or D is adjacent to C. If B were on an end of the line, D would either have to be adjacent to C, in which case F could not be between two boys, adjacent to B, in which case F could not be between two boys, or on the other end, in which case F could not be between two boys. Therefore, B and D are adjacent to each other and they are surrounded by two girls. It follows that F is adjacent to C and one of B or D. From here, we can find that the only possible permutations are $ABDFCE, ADBFCE, ACFBDE$, and $ACFDBE$, for a total of $\boxed{4}$ possibilities.

6. A *spirited* integer is a positive number representable in the form 20^n+13k for some positive integer n and any integer k. Determine how many *spirited* integers are less than 2013.

Solution: Let the smallest positive integer x such that $20^x \equiv 1 \pmod{13}$ be the order of 20 (mod 13). This is the same as the order of 7 (mod 13), as $20^x \equiv 7^x \pmod{13}$ for all x. Because 7 is relatively prime to 13, we must have that the order of 7 (mod 13) is a factor of $\phi(13) = 12$. Listing out the remainders when powers of 7 are divided by 13, we get $7, 10, 5, 9, 11, 3, \cdots$. Because we have already listed 6 powers of 7 without encountering one with a remainder of 1, we know that the order of 7 (mod 13) or the order of 20 (mod 13) is 12. This means that the powers of 20 can take any remainder when divided by 13 except 0. Therefore, the only numbers which are not representable in the form $20^n + 13k$ are multiples of 13. It follows that our answer is $2012 - \lfloor \frac{2012}{13} \rfloor = 2012 - 154 = \boxed{1858}$.

7. A finite set of distinct, nonnegative integers $\{a_1, ..., a_k\}$ is called *admissible* if the integer function $f(n) = (n+a_1)\cdots(n+a_k)$ has no common divisor over all terms; that is, $\gcd(f(1), f(2), ..., f(n)) = 1$ for any integer n. How many *admissible* sets only have members of value less than 10? $\{4\}$ and $\{0, 2, 6\}$ are such sets, but $\{4, 9\}$ and $\{1, 3, 5\}$ are not.

Solution: If $\gcd(f(1), f(2), ..., f(n)) = 1$ for any value of n, then it suffices for $\gcd(f(1), f(2))$ to be equal to 1. It follows that $\gcd((1 + a_1)(1 + a_2)\cdots(1 + a_k), (2 + a_1)(2 + a_2)\cdots(2 + a_k)) = 1$. It follows that for any prime p, there cannot be two values in a_k a_i, a_j such that $a_i \equiv p - 2 \pmod{p}$ and $a_j \equiv p - 1 \pmod{p}$. This gives us two cases:

Case 1: All members of a_k are odd.

In this case, $a_k \in \{1, 3, 5, 7, 9\}$. We know that we cannot have an element which leaves a remainder of 1 (mod 3) and also have an element which leaves a remainder of 2 (mod 3). This means we cannot have a value from $\{1, 7\}$ while also having a value from $\{5\}$. Similarly, we cannot have an element which leaves a remainder of 3 (mod 5) and also have an element which leaves a remainder of 4 (mod 5). It follows that we cannot have both 3 and 9. This means that the total number of possibilities in this case is $3 \cdot 5 - 1 = 14$.

Case 2: All members of a_k are even.

In this case, $a_k \in \{0, 2, 4, 6, 8\}$. We know that we cannot have an element which leaves a remainder of 1 (mod 3) and also have an element which leaves a remainder of 2 (mod 3), so we cannot have a value from $\{4\}$ while also having a value from $\{2, 8\}$. Similarly, we cannot have an element which leaves a remainder of 3 (mod 5) and also have an element which leaves a remainder of 4 (mod 5). Therefore, we cannot have both 8 and 4. It follows that the total number of possibilities for what elements from the set $\{2, 4, 8\}$ which work is 5. It does not matter if $\{0, 6\}$ are in the set, so the total number of possibilities if $5 \cdot 4 - 1 = 19$.

Adding up our cases and the empty set, we have a total of $17 + 23 + 1 = \boxed{34}$ possibilities.

Note: According to the Berkeley Math Tournament solution for this problem, the answer is 44. However, they over-counted several possibilities. I have checked my answer with the following Python code:

```
import fractions
def getSubsets(x):
    if(len(x)==1):
        return [x, []]
    a = getSubsets(x[1:])
    t = []
    for y in a:
        t.append(y)
```

```
        c = []
        for z in y:
            c.append(z)
        c.append(x[0])
        t.append(c)
    return t

d = getSubsets([0,1,2,3,4,5,6,7,8,9])

def prod(a, b):
    if(len(b)==0):
        return 1
    t = 1
    for x in b:
        t*=(a+x)
    return t

c = 0

for x in d:
    if(fractions.gcd(prod(1,x),prod(2,x)) == 1):
        c+=1

print c
```

This code gives an output of 34.

8. The pages of a book are consecutively numbered from 1 through 480. How many times does the digit 8 appear in this numbering?

Solution: We will do casework on the place in our page numbers where the digit 8 appears.

Case 1: The tens digit is an 8.

When the hundreds digit is 3 or less, there are ten possible units digits in this case. This gives us a total of $4 \cdot 10 = 40$ numbers. When the hundreds digit is 4, the units digit has to be 0, so in total we have $40 + 1 = 41$ numbers in this case.

Case 2: The units digit is an 8.

This will occur once in every group of 10 from $0 - 9$ to $470 - 479$. There are a total of 48 of these groups, so we have 48 numbers in this case.

Adding up our cases, we have a total of $41 + 48 = \boxed{89}$ numbers in this case.

9. A positive integer is said to be *binary-emulating* if its base three representation consists of only 0s and 1s. Determine the sum of the first 15 *binary-emulating* numbers.

Solution: Notice that the nth *binary-enulating* number is the result when the digits of n written in binary are interpreted in base 3 and converted back to base 10. The first 15 *binary-emulating* numbers therefore, are the numbers 0001_3, 0010_3, $0011_3 \cdots 1111_3$. It follows that the sum of these numbers is $(1 + 3 + 9 + 27) \cdot 8 = \boxed{320}$.

10. Professor X can choose to assign homework problems from a set of problems labeled 1 to 30, inclusive. No two problems in his assignment can share a common divisor greater than 1. What is the maximum number of problems that Professor X can assign?

Solution: Notice that there are 10 primes less than 30 $(2, 3, 5, 7, 11, 13, 17, 19, 23, 29)$. It follows that the largest set of problems that Professor X can assign is a set similar to $\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$, as if any more problems were added, there would be two problems with the same prime factor. It follows that the largest number of problems that Professor X can assign is $\boxed{11}$.

11. Alice and Bob play a game of Berkeley Ball, in which the first person to win four rounds is the winner. No round can end in a draw. How many distinct games can be played in which Alice is the winner? (Two games are said to be identical if either player wins/loses rounds in the same order in both games.)

Solution: Notice that Alice must win the last game, and she must win 3 games before that while Bob wins less than or equal to 3 games before that. Therefore, our answer is $\binom{3+0}{3} + \binom{3+1}{3} + \binom{3+2}{3} + \binom{3+3}{3} = \binom{7}{4} = \boxed{35}$ by the Hockey Stick Identity.

12. How many ways are there to place 3 indistinguishable pegs on a 5×5 chessboard so that no two pegs lie in the same row or column?

Solution: First assume the pegs are distinguishable. There are 25 ways to choose the location of the first peg. From here, one row and one column are eliminated, leaving 16 ways to choose the location of the second peg. Finally, there are 9 ways to choose the location of the third peg. Therefore, when the pegs are distinguishable, the number of ways to choose their locations is $25 \cdot 16 \cdot 9$. However, because the pegs are indistinguishable, we must divide by $3! = 6$ to account for equivalent positions with different orderings of the pegs. Therefore, our answer is $\frac{25 \cdot 16 \cdot 9}{6} = \boxed{600}$.

13. Two boxes contain some number of red, yellow, and blue balls. The first box has 3 red, 4 yellow, and 5 blue balls, and the second box has 6 red, 2 yellow, and 7 blue balls. There are two ways to select a ball from these boxes; one could first randomly choose a box and then randomly select a ball or one could put all the balls in the same box and simply randomly select a ball from there. How much greater is the probability of drawing a red ball using the second method than the first?

Solution: In the first method of choosing a ball, the probability of selecting a red ball is $\frac{1}{2} \cdot \frac{3}{12} + \frac{1}{2} \cdot \frac{6}{15} = \frac{13}{40}$. In the second method of choosing a ball, the probability of selecting a red ball is $\frac{3+6}{12+15} = \frac{1}{3}$. Therefore, the difference is $\frac{1}{3} - \frac{13}{40} = \boxed{\frac{1}{120}}$.

14. A coin is flipped until there is a head followed by two tails. What is the probability that this will take exactly 12 flips?

Solution: Let H_n be the number of ways to first get a head followed by two tails in n flips. If we let H represent heads and T represent tails, then a permutation with this property and n flips is equivalent to appending HTT to the end of a permutation without this property and $n - 3$ flips. The total number of permutations with $n - 3$ flips is 2^{n-3}, and the number of permutations with the property and $n-3$ flips is $H_{n-3} + 2H_{n-4} + 4H_{n-5} + \cdots$. Therefore, $H_n = 2^{n-3} - H_{n-3} - 2H_{n-4} - 4H_{n-5} - \cdots$ for $n \geq 3$. We know $H_0 = 0, H_1 = 0$, and $H_2 = 0$, so using this recursion repeatedly we get $H_{12} = 143$. The total number of possibilities in 12 flips is $2^{12} = 4096$, so our answer is $\boxed{\frac{143}{4096}}$.

15. If I roll three fair 4-sided dice, what is the probability that the sum of the resulting numbers is relatively prime to the product of the resulting numbers?

Solution: If we let the rolls be x, y, z, then this property is equivalent to $\gcd(x, y+z) = 1$, $\gcd(y, x+z) = 1$, and $\gcd(z, x+y) = 1$. By brute force, we can find that the only triples (x, y, z) that satisfy this are $(1,1,1), (1,1,3), (1,2,2), (1,2,4), (1,3,3), (1,4,4), (2,2,3), (3,4,4)$ and permutations. Adding up all of these permutations, we get a total of $1+3+3+6+3+3+3+3 = 25$ triples which work. The total number of possibilities when 3 4-sided dice are rolled is $4^3 = 64$. Therefore, our answer is $\boxed{\frac{25}{64}}$.

3 Sources

1. Berkeley Math Tournament Individual Fall 2013 Problem 6
2. Berkeley Math Tournament Individual Fall 2013 Problem 8
3. Berkeley Math Tournament Individual Fall 2013 Problem 10
4. Berkeley Math Tournament Individual Fall 2013 Problem 12
5. Berkeley Math Tournament Individual Fall 2013 Problem 13
6. Berkeley Math Tournament Individual Fall 2013 Problem 18
7. Berkeley Math Tournament Individual Fall 2013 Problem 20
8. Berkeley Math Tournament Team Fall 2013 Problem 8
9. Berkeley Math Tournament Team Fall 2013 Problem 12
10. Berkeley Math Tournament Team Fall 2013 Problem 13
11. Berkeley Math Tournament Team Fall 2013 Problem 15
12. Berkeley Math Tournament Speed Fall 2013 Problem 59
13. Berkeley Math Tournament Individual Spring 2013 Problem 3
14. Berkeley Math Tournament Discrete Spring 2013 Problem 6
15. Berkeley Math Tournament Discrete Spring 2013 Problem 2