Combinatorics Handout #6 Answers and Solutions

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1 Answers

1. $\dfrac{400}{9}$

2. 6435

3. 30

4. 3744

5. 56

6. 44

7. $\dfrac{10201}{100}$

8. 420

9. 6

10. $\dfrac{3}{5}$

11. 1001

12. 34

13. $\dfrac{231}{1024}$

14. 21

15. 52

2 Solutions

1. On 5×5 grid of lattice points, every point is uniformly randomly colored blue, red, or green. Find the expected number of monochromatic triangles T with vertices chosen from the lattice grid, such that some two sides of T are parallel to the axis.

Solution: The probability that an arbitrary triangle will be monochromatic is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ as the second and third colors of the vertices must match the color of the first vertex. The total number of triangles with two sides parallel to the axes is $25 \cdot 4 \cdot 4 = 400$ as there are 25 ways to choose which vertex will have a right angle, 4 ways to choose which vertex will be in the same row as the first, and 4 ways to choose which vertex will be in the same column as the first. Because each point appears in the same number of these triangles, the expected number of monochromatic triangles is $400 \cdot \frac{1}{9} = \boxed{\dfrac{400}{9}}$.

2. What is the number of nondecreasing positive integer sequences of length 7 whose last term is at most 9?

Solution: Let the number of nondecreasing positive integer sequences of length n and first element k with a last term of at most 9 be $F_{n,k}$. It follows that

$$F_{n,k} = \sum_{i=k}^{9} F_{n-1,i}$$

where $F_{n,9} = 1$ for all n and $F_{1,k} = 1$ for all k. Using these facts, we can generate the following grid:

n/k	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	9	8	7	6	5	4	3	2	1
3	45	36	28	21	15	10	6	3	1
4	165	120	84	56	35	20	10	4	1
5	495	330	210	126	70	35	15	5	1
6	1287	792	462	252	126	56	21	6	1
7	3003	1716	924	462	210	84	28	7	1

Adding up the elements in the 7th row, we get an answer of $\binom{15}{7} = \boxed{6435}$.

3. How many ways can we pick four 3-element subsets of $\{1, 2, ..., 6\}$ so that each pair of subsets share exactly one element?

Solution: Let the first subset be A, B, C. Then without loss of generality, let the second subset be A, D, E. Then let the third subset be B, D, F. It follows that the fourth subset is C, E, F. If we consider some arbitrary element A, then there are 5 ways to choose the element F which does not appear in the same subset as A and 3 ways to choose which pairs of elements appear in the same subset as A. In addition, there are 2 ways to choose the pairs of elements that are in the same subset as F. Following all of these choices, there is exactly one set of four subsets with these properties. It follows that our answer is $5 \cdot 3 \cdot 2 \cdot 1 = \boxed{30}$.

4. How many five-card hands from a standard deck of 52 cards are full houses? A full house consists of 3 cards of one rank and 2 cards of another rank.

Solution: There are 13 ways to choose which rank has 3 cards and 12 ways to choose which rank has 2 cards. There are $\binom{4}{3} = 4$ ways to choose which 3 cards are of the first rank and $\binom{4}{2} = 6$ ways to choose which 6 cards are of the second rank. It follows that our answer is $13 \cdot 12 \cdot 4 \cdot 6 = \boxed{3744}$.

5. Three 3-legged (distinguishable) Stanfurdians take off their socks and trade them with each other. How many ways is this possible if everyone ends up with exactly 3 socks and nobody gets any of their own socks? All socks originating from the Stanfurdians are distinguishable from each other. All Stanfurdian feet are indistinguishable from other feet of the same Stanfurdian.

Solution: Label the socks $1, 2, 3, 4, 5, 6, 7, 8, 9$ where the first three are from the first Stanfurdian and the next three are from the second Stanfurdian. Then we want to regroup these numbers such that no sock is in its original group of 3. First consider if the three Stanfurdians simply cycle each of their three socks. In other words, if the Stanfurdians are A, B, and C, then consider when A

gives his socks to B, B gives his socks to C, and C gives his socks to A. Clearly there are 2 ways in which this is possible. The only other case is when A gives 2 of his socks to B and the other sock to C, B gives 2 of his socks to C and the other sock to A, and C gives 2 of his socks to A and the other sock to B. In this case, there are 2 ways to choose which Stanfurdian that A gives 2 socks, 3 ways to choose which pair of socks that A gives, 3 ways to choose which pair of socks that B gives, and 3 ways to choose which pair of socks that C gives. It follows that there are $2 \cdot 3 \cdot 3 \cdot 3 = 54$ possibilities in this case. It follows that our answer is $2 + 54 = \boxed{56}$.

6. How many subsets (including the empty-set) of $\{1, 2..., 6\}$ do not have three consecutive integers?

Solution: In total there are $2^6 = 64$ subsets of this set. $2^{6-3} = 8$ of them will have $1, 2,$ and 3. $2^{6-4} = 4$ of them will have $2, 3,$ and 4 but not 1. $2^{6-5} \cdot 2^1 = 4$ of them will have $3, 4,$ and 5 but not 2. $2^{6-6} \cdot 2^2 = 4$ of them will have $4, 5,$ and 6 but not 3. It follows that in total, there are $8 + 4 + 4 + 4 = 20$ subsets with three consecutive integers, and it follows that there are a total of $64 - 20 = \boxed{44}$ subsets with this property.

7. Consider an urb containing 51 white and 50 black balls. Every turn, we randomly pick a ball, record the color of the ball, and then we put the ball back into the urn. We stop picking when we have recorded n black balls, where n is an integer randomly chosen from $\{1, 2, ..., 100\}$ What is the expected number of turns?

Solution: Notice that the probability that any given ball is black is $\frac{50}{50+51} = \frac{50}{101}$. It follows that the expected number of black balls after x choices is $\frac{50x}{101}$. It follows that the expected number of turns before n black balls are drawn is $\frac{n}{\frac{50}{101}} = \frac{101n}{50}$. The expected value of n is $\frac{1+100}{2} = \frac{101}{2}$, so our answer is $\frac{101}{50} \cdot \frac{101}{2} = \boxed{\dfrac{10201}{100}}$.

8. There are 2000 math students and 4000 CS students at Berkeley. If 5580 students are either math students or CS students, then how many of them are studying both math and CS?

Solution: Notice that by adding the 2000 math students to the 4000 CS students, we will count 6000 students, but we will have counted each of the students in both math and CS exactly twice. It follows that the number of students who are studying both math and CS is $6000 - 5580 = \boxed{420}$.

9. I have 18 socks in my drawer, 6 colored red, 8 colored blue and 4 colored green. If I close my eyes and grab a bunch of socks, how many socks must I grab to guarantee there will be two pairs of matching socks?

Solution: Notice that to maximize the number of socks we can take before we encounter two pairs of matching socks, we should choose 3 of one color of sock so that we only get 1 pair of that color, and then we should take 1 sock of each of the other colors. Once we choose our 6th sock, we must have 2 pairs of socks regardless of the color of that sock, so our answer is $\boxed{6}$.

10. Alice, Bob, and four other people sit themselves around a circular table. What is the probability that Alice does not sit to the left or right of Bob?

Solution: Clearly there are 3 seats which are not next to Bob and 5 for Alice to choose from. It follows that our answer is $\boxed{\dfrac{3}{5}}$.

11. There are 20 indistinguishable balls to be placed into bins $A, B, C, D,$ and E. Each bin must have at least 2 balls inside of it. How many ways can the balls be placed into the bins, if each ball must be placed in a bin?

Solution: Notice that once 2 balls are placed in each bin, we wish to divide the remaining 10 balls among 5 bins such that there is a nonnegative number of balls in each bin. By Stars and Bars, it follows that our answer is $\binom{10+4}{4} = \boxed{1001}$.

12. A *gorgeous* sequence is a sequence of 1's and 0's such that there are no consecutive 1's. For instance, the set of all *gorgeous* sequences of length 3 is $\{[1,0,0],[1,0,1],[0,1,0],[0,0,1],[0,0,0]\}$. Determine the number of *gorgeous* sequences of length 7.

Solution: Let F_n be the number of *gorgeous* sequences of length n. Notice that if the nth digit in such a sequence were 1, then by definition the $n-1$th digit would have to be a 0 and then the first $n-2$ digits would have to be any arbitrary *gorgeous* sequence. Similarly, if the nth digit in such a sequence were 0, then the first $n-1$ digits would have to be any arbitrary *gorgeous* sequence. It follows that $F_n = F_{n-1} + F_{n-2}$. By brute force we can observe that $F_1 = 2$ and $F_2 = 3$. It follows that $F_3 = 5, F_4 = 8, F_5 = 13, F_6 = 21$, and finally $F_7 = \boxed{34}$.

13. A coin is flipped 12 times. What is the probability that the total number of heads equals the total number of tails? Express your answer as a common fraction in lowest terms.

Solution: If the total number of heads is equal to the total number of tails, then it follows that we must have exactly 6 heads and exactly 6 tails. The total number of ways to achieve this is $\binom{12}{6} = 924$. The total number of possibilities when a coin is flipped 12 times is $2^{12} = 4096$. It follows that our answer is $\frac{924}{4096} = \boxed{\dfrac{231}{1024}}$.

14. Debbie has six Pusheens: 2 pink ones, 2 gray ones, and 2 blue ones, where Pusheens of the same color are indistinguishable. She sells two Pusheens each to Alice, Bob, and Eve. How many ways are there for her to do so?

Solution: There are 3 cases. The first case is that each person takes 2 Pusheens of the same color. Clearly in this case there are $3! = 6$ possibilities. The second case is that one person takes 2 Pusheens of the same color and the other two people take 1 Pusheen of 1 color and 1 Pusheen of the other color. In this case there are 3 ways to choose who takes 2 Pusheens of the same color and 3 ways to choose which color they take from. It follows that there are $3^2 = 9$ possibilities in this case. The third case is that each person takes 1 Pusheen of one color and 1 Pusheen of another color. There are 3 ways to choose which color Alice does not take and then there are 2 ways to choose which color Bob does not take. It follows that there are $3 \cdot 2 = 6$ possibilities in this case. It follows that our answer is $6 + 9 + 6 = \boxed{21}$.

15. A *prim* number is a number that is prime if its last digit is removed. A *rime* number is a number that is prime if its first digit is removed. Determine how many numbers between 100 and 999 inclusive are both *prim* and *rime* numbers.

Solution: Notice that if our number is of the form $\overline{abc}$, we must have that b is odd because the number $\overline{ab}$ is prime and greater than 10 and we must have that c is odd because the number $\overline{bc}$ is prime and greater than 10. Therefore, $\overline{bc}$ is a 2-digit prime number with two odd digits which are both not equal to 5. It follows that the possible values of $\overline{bc}$ are $\{11, 13, 17, 19, 31, 37, 71, 73, 79, 97\}$. Now we can notice that there are 5 2-digit prime numbers that end in 1, 6 2-digit prime numbers that end in 3, 5 2-digit prime numbers that end in 7, and 5 2-digit prime numbers that end in 9. It follows that our answer is $4 \cdot 5 + 2 \cdot 6 + 3 \cdot 5 + 1 \cdot 5 = \boxed{52}$.

3 Sources

1. 2016 Berkeley Math Tournament Spring Individual Problem 9
2. 2016 Berkeley Math Tournament Spring Individual Problem 12
3. 2016 Berkeley Math Tournament Spring Individual Problem 15
4. 2016 Berkeley Math Tournament Spring Discrete Problem 3
5. 2016 Berkeley Math Tournament Spring Discrete Problem 4
6. 2016 Berkeley Math Tournament Spring Team Problem 9
7. 2016 Berkeley Math Tournament Spring Team Problem 13
8. 2017 Berkeley Math Tournament Fall Individual Problem 3
9. 2017 Berkeley Math Tournament Fall Individual Problem 8
10. 2017 Berkeley Math Tournament Fall Individual Problem 13
11. 2017 Berkeley Math Tournament Fall Individual Problem 18
12. 2017 Berkeley Math Tournament Fall Individual Problem 20
13. 2017 Berkeley Math Tournament Fall Team Problem 5
14. 2017 Berkeley Math Tournament Fall Team Problem 8
15. 2017 Berkeley Math Tournament Fall Team Problem 12