Combinatorics Handout # 7 Answers and Solutions

Walker Kroubalkian

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1 Answers

1. $\frac{1}{11}$

2. 36

3. 89

4. 1296

5. 34

6. $\frac{4}{9}$

7. 1997

8. 2063

9. 11

10. 34

11. $\frac{2019}{1001}$

12. 201

13. $\frac{3}{2017}$

14. 12

15. 225

2 Solutions

1. You have 9 colors of socks and 5 socks of each type of color. Pick two socks randomly. What is the probability that they are the same color?

Solution: Consider an arbitrary sock. There are $9 \cdot 5 - 1 = 44$ socks which are not this sock, and only $5 - 1 = 4$ of them have the same color. Therefore, our answer is $\frac{4}{44} = \boxed{\frac{1}{11}}$.

2. Each BMT, every student chooses one of three focus rounds to take. Bob plans to attend BMT for the next 4 years and wants to figure out what focus round to take each year. Given that he wants to take each focus round at least once, how many ways can he choose which round to take each year?

Solution: There are 3 ways to choose which Focus Round Bob takes twice, $\binom{4}{2} = 6$ ways to choose which two years Bob takes that Focus Round, and 2 ways to choose which order Bob takes the other two Focus Rounds. It follows that our answer is $3 \cdot 6 \cdot 2 = \boxed{36}$.

3. How many subsets of $\{1, 2, ..., 9\}$ do not contain 2 adjacent numbers?

Solution: Let F_n be the number of subsets of $\{1, 2, ..., n\}$ which do not contain 2 adjacent numbers. When $n \geq 3$, we claim that $F_n = F_{n-1} + F_{n-2}$. This is because if n is included in a subset, then the remaining elements must come from some subset included in F_{n-2} whereas when n is not included in a subset, the remaining elements must come from subset included in F_{n-1}. We know that $F_1 = 2$ and $F_2 = 2^2 - 1 = 3$. Therefore, by brute force we can find that the next few terms of this sequence are $5, 8, 13, 21, 34, 55$, and finally $F_9 = \boxed{89}$.

4. Let $S = \{1, 2, ...6\}$. How many functions $f : S \to S$ are there such that for all $s \in S$,

$$f^5(s) = f(f(f(f(f(s))))) = 1.$$

Solution: We can begin by noting that $f(1) = 1$, as if $f(1) = n$ for some $n \neq 1$, then we would have that $f^5(n) = f(f^5(1)) = f(1) = n \neq 1$. If we consider this function as a directed graph in graph theory where each input represents a vertex of our graph and each edge extends from the input to its respective output, then it follows that the graph is a tree with a base at the vertex 1. From here, we can do casework on the branches of this tree. If the tree has one branch with 5 vertices extending from the base, then there are $5! + 5! + 5! + \frac{5!}{2!} + \frac{5!}{3!} + \frac{5!}{2!} + \frac{5!}{2!} + \frac{5!}{4!} + \frac{5!}{2!} = 625$ possibilities. If the tree has one branch with 4 vertices extending from the base and one branch with 1 vertex extending from the base, then there are $5! + \frac{5!}{2!} + 5! + \frac{5!}{3!} = 320$ possibilities. If the tree has one branch with 3 vertices extending from the base and one branch with 2 vertices extending from the base, then there are $5! + \frac{5!}{2!} = 180$ possibilities. If the tree has one branch with 3 vertices extending from the base, and two branches each with one vertex extending from the base, then there are $\frac{5!}{2!} + \frac{5!}{2! \cdot 2!} = 90$ possibilities. If the tree has one branch with 1 vertex extending from the base and two branches each with 2 vertices extending from the base, then there are $\frac{5!}{2!} = 60$ possibilities. If the tree has one branch with 2 vertices extending from the base and 3 branches each with 1 vertex extending from the base, then there are $\frac{5!}{3!} = 20$ possibilities. If the tree has 5 branches each with 1 vertex extending from the base, then there is $\frac{5!}{5!} = 1$ possibility. Therefore our answer is that there are $625 + 320 + 180 + 90 + 60 + 20 + 1 = \boxed{1296}$ possible functions.

Note: While this method confirms the answer key that the Berkeley Math Tournament provided, it raises a different question. Is there a good reason why the answer is $6^4 = 1296$? There might be, but the Berkeley Math Tournament did not provide a solution, so I am not sure why this may be true at the moment.

5. You enter an elevator on floor 0 of a building with some other people, and request to go to floor 10. In order to be efficient, it doesn't stop at adjacent floors (so, if it's at floor 0, its next stop cannot be floor 1). Given that the elevator will stop at floor 10, no matter what other floors it stops at, how many combinations of stops are there for the elevator?

Solution: Let F_n be the number of ways to go up n floors without stopping on adjacent floors. We claim that $F_n = F_{n-2} + F_{n-1}$. This is because if we stop at the floor 2 above our start, then the remaining moves must come from one of the possibilities in F_{n-2} whereas if we don't stop at the floor 2 above our start, then the question is equivalent to going up $n - 1$ floors. We know that $F_1 = 0$ and $F_2 = 1$. It follows that the next few terms of this sequence are $1, 2, 3, 5, 8, 13, 21$, and finally $F_{10} = \boxed{34}$.

Note: The answer key states that 55 is also an acceptable answer. I don't know what interpretation would produce this answer at the moment.

6. Eric has a 9-sided die and Harrison has an 11-sided die. They each roll their respective die. Eric wins if he rolls a number greater than or equal to Harrison's number. What is the probability that Eric wins?

Solution: If Eric rolls $n \geq 2$, then there are n possibilities for what Harrison could roll such that Eric would have a higher number. It follows that there are $\sum_{n=2}^{9} = 44$ possibilities where Eric wins. It follows that our answer is $\frac{44}{9 \cdot 11} = \boxed{\dfrac{4}{9}}$.

7. Ed writes the first 2018 positive integers down in order: $1, 2, 3, ..., 2018$. Then for each power of 2 that appears, he crosses out that number as well as the number 1 greater than that power of 2. After he is done, how many numbers are not crossed out?

Solution: There are 11 powers of 2 less than 2018. These include $2^0, 2^1, ..., 2^9$, and 2^{10}. It follows that for each of these powers of 2, we will cross out 2 numbers. However, we will cross out 2^1 twice as $2^1 = 2^0 + 1$. Therefore, we will cross out $2 \cdot 11 - 1 = 21$ numbers. It follows that our answer is $2018 - 21 = \boxed{1997}$.

8. The sequence $2, 3, 5, 6, 7, 8, 10, ...$ contains all positive integers that are not perfect squares. Find the 2018th term of the sequence.

Solution: There are 44 perfect squares that are less than or equal to 2018. It follows that the 2018th term is at least $2018 + 44 = 2062$. However, $45^2 = 2025$ is between 2018 and 2062, so our answer is actually $\boxed{2063}$.

9. A number is formed using the digits $\{2, 0, 1, 8\}$, using all 4 digits exactly once. Note that $0218 = 218$ is a valid number that can be formed. What is the probability that the resulting number is strictly greater than 2018?

Solution: There are $4! = 24$ possible rearrangements. If the thousands digit is 8, then the resulting number has to be greater than 2018. If the thousands digit is 2, the the number is at least equal to 2018 and is only exactly equal in 1 case. Therefore, our answer is $2 \cdot 3! - 1 = \boxed{11}$.

10. A dice is labeled with the integers $1, 2, ..., n$ such that it is 2 times as likely to roll a 2 as it is a 1, 3 times as likely to roll a 3 as it is a 1, and so on. Suppose the probability of rolling an odd integer with the dice is $\frac{17}{35}$. Compute n.

Solution: Notice that the denominator of the probability of any result is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. The numerator of the probability that an odd result is rolled has to be a perfect square as $1 + 3 + \cdots + (2x - 1) = x^2$ where x is roughly half of n. Notice that if $n = 2 \cdot 17 = 34$, then every condition is satisfied as $\frac{17^2}{\frac{34 \cdot 35}{2}} = \frac{17}{35}$. It follows that our answer is $n = \boxed{34}$.

11. Let S be the set of all 1000 element subsets of the set $\{1, 2, 3, ..., 2018\}$. What is the expected value of the minimum element of a set chosen uniformly at random from S?

Solution: There are $\binom{2017}{999}$ subsets with a minimum element of 1, $\binom{2016}{999}$ subsets with a minimum element of 2, and so on. It follows that the sum of the minimum elements of all subsets is

$$\sum_{n=1}^{1019} n \cdot \binom{2018 - n}{999} = \binom{2017}{999} + 2 \cdot \binom{2016}{999} + 3 \cdot \binom{2015}{999} + \cdots + 1019 \cdot \binom{999}{999}$$

By the Hockey Stick Identity, we know that $\binom{n}{k}+\binom{n-1}{k}+\binom{n-2}{k}+\cdots+\binom{k}{k}=\binom{n+1}{k+1}$. It follows that the given sum is equivalent to the sum

$$\sum_{n=1}^{1019} n\cdot\binom{2018-n}{999}=\sum_{n=1}^{1019}\binom{2019-n}{1000}=\binom{2018}{1000}+\binom{2017}{1000}+\cdots+\binom{1000}{1000}=\binom{2019}{1001}$$

The total number of subsets is $\binom{2018}{1000}$. It follows that our answer is $\frac{\binom{2019}{1001}}{\binom{2018}{1000}}=\boxed{\frac{2019}{1001}}$.

12. A lattice point is a point (a, b) on the Cartesian plane where a and b are integers. Compute the number of lattice points in the interior and on the boundary of the triangle with vertices at $(0, 0), (0, 20)$, and $(18, 0)$.

Solution: Notice that there are $21 \cdot 19 = 399$ lattice points on the boundary or in the interior of the rectangle with corners at $(0, 0), (0, 20), (18, 0)$ and $(18, 20)$. Roughly half of these will lie in the given triangle. However, the points on the diagonal side of this triangle will lie in both triangles, so we must account for that. There are 3 lattice points on the segment connecting $(18, 0)$ and $(0, 20)$. It follows that our answer is $\frac{399-3}{2}+3=\boxed{201}$.

13. Three distinct points are chosen uniformly at random from the vertices of a regular 2018-gon. What is the probability that the triangle formed by these points is a right triangle?

Solution: A triangle will be a right triangle if its longest side is one of the longest diagonals of the regular 2018-gon. There are 1009 longest diagonals in a regular 2018-gon. There are 2016 points which can be chosen as a third point to complete a right triangle for a total of $1009 \cdot 2016$ right triangles. There are $\binom{2018}{3}$ total triangles. It follows that our answer is $\frac{1009\cdot 2016}{\binom{2018}{3}}=\frac{1009\cdot 2016\cdot 6}{2018\cdot 2017\cdot 2016}=\boxed{\frac{3}{2017}}$.

14. How many ways are there to partition 11 into a sum of an odd number of odd positive integers? Order does not matter, so $11 = 3 + 3 + 5$ and $11 = 3 + 5 + 3$ should be counted only once.

Solution: Doing casework on the largest numbers in our partitions, we find that the only partitions are $11, 9+1+1, 7+3+1, 7+1+1+1+1, 5+5+1, 5+3+3, 5+3+1+1+1, 5+1+1+1+1+1+1, 3+3+3+1+1, 3+3+1+1+1+1+1, 3+1+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1+1+1$. It follows that our answer is $\boxed{12}$.

15. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

Solution: The minimum number of distinct values will occur when most elements except the unique mode appear in the list 9 times. This results in $1+\lceil\frac{2008}{9}\rceil=\boxed{225}$.

3 Sources

1. 2017 Berkeley Math Tournament Spring Discrete Problem 1
2. 2017 Berkeley Math Tournament Spring Discrete Problem 2
3. 2017 Berkeley Math Tournament Spring Discrete Problem 5

4. 2017 Berkeley Math Tournament Spring Discrete Problem 6
5. 2017 Berkeley Math Tournament Spring Team Problem 5
6. 2018 Stanford Math Tournament Spring General Problem 6
7. 2018 Stanford Math Tournament Spring General Problem 9
8. 2018 Stanford Math Tournament Spring General Problem 11
9. 2018 Stanford Math Tournament Spring Discrete Problem 1
10. 2018 Stanford Math Tournament Spring Discrete Problem 3
11. 2018 Stanford Math Tournament Spring Discrete Problem 7
12. 2018 Stanford Math Tournament Spring Team Problem 3
13. 2018 Stanford Math Tournament Spring Team Problem 6
14. 2018 Stanford Math Tournament Spring General Tiebreaker Problem 3
15. 2018 AMC 12B Problem 10