Combinatorics Handout 1 Answers and Solutions

Walker Kroubalkian

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1 Answers

1. $\frac{1}{16}$

2. $\frac{2}{5}$

3. $\frac{2}{3}$

4. 100

5. $365\frac{97}{400}$

6. 30

7. 56

8. 4

9. 2880

10. $\frac{11}{5}$

11. 13

12. 10

13. 277

14. 1242

15. 872

2 Solutions

1. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

Solution: In order for the fourth coin flip to be the first head, the first three coin flips must all be tails, meaning that our first four flips must be TTTH (T is Tails, H is heads). The total number of possibilities for the first four coin flips is $2^4 = 16$, and only one of these is TTTH, so our probability is $\boxed{\dfrac{1}{16}}$.

2. Ted has 2 red socks, 2 blue socks, and 2 green socks. He grabs three different socks at random. Compute the probability that they are all different colors.

Solution: Notice that no matter what color Ted gets for his first sock, there are only 4 socks which remain of a different color. Therefore, the probability that the second sock is a different color from the first is $\frac{4}{5}$, and by similar logic the probability that the third sock is different from the other two is $\frac{2}{4} = \frac{1}{2}$. Therefore our answer is $\frac{4}{5} \times \frac{1}{2} = \boxed{\frac{2}{5}}$.

3. Alice and Bob are playing a game in which Alice has a $\frac{1}{3}$ probability of winning, a $\frac{1}{2}$ probability of tying, and a $\frac{1}{6}$ probability of losing. Given that Alice and Bob played a game which did not end in a tie, compute the probability that Alice won.

Solution: The probability that a given game does not end in a tie is $1 - \frac{1}{2} = \frac{1}{2}$. The probability that Alice wins is $\frac{1}{3}$, so our final probability is $\frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$. Notice that Alice cannot win when the game ends in a tie, so her entire likelihood of winning is included in the numerator of our fraction.

4. Let a 5-digit number be termed a *valley* number if the digits (not necessarily distinct) in the number $\overline{abcde}$ satisfy $a > b > c$ and $c < d < e$. Compute the number of *valley* numbers that start with 3.

Solution: We will organize the *valley* numbers into two different cases based on the value of c:
Case 1: $c = 1$
By the definition of a *valley* number, when $c = 1$ and $a = 3$, $b = 2$. Therefore we only need to worry about the values of d and e. We must have $e > d > 1$. It follows that d and e must come from the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$. Without the restriction that $e > d$, there are 8 ways to choose the value of d and 7 ways to choose a remaining value for e, for a total of $8 \times 7 = 56$ possibilities. By symmetry, half of these will have $e > d$. Therefore there are $\frac{56}{2} = 28$ possibilities for this case.
Case 2: $c = 0$
Because $a > b > c$, we have b is among the set $\{1, 2\}$. b is independent of our choices for d and e, so we can just double the number of ways to choose d and e. Because $0 < d < e$, we must have d and e come from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Without the restriction that $e > d$, there are 9 ways to choose the value of d and 8 ways to choose a remaining value for e, for a total of $9 \times 8 = 72$ possibilities. By symmetry, half of these will have $e > d$. Therefore, there are $\frac{72}{2} = 36$ ways to choose d and e and in total there are $2 \times 36 = 72$ possibilities for this case.
Adding our cases together, we have a total of $28 + 72 = \boxed{100}$ *valley* numbers with this property.

5. Including the leap years, compute the average number of days in a year. Express your answer as a mixed number. (Years that are evenly divisible by 4 are leap years. However years that are evenly divisible by 100 are not leap years, unless they are also evenly divisible by 400, in which case they are leap years.)

Solution: Notice that by the definition a leap year, leap years will repeat in frequency every $lcm(4, 100, 400) = 400$ years. Therefore, we only need to consider each group of 400 years. Notice that every 400 years there are $\frac{400}{4} = 100$ years which are multiples of 4. However, this includes multiples of 100, so we must subtract $\frac{400}{100} = 4$. However, this does not include multiples of 400, so we must add $\frac{400}{400} = 1$. Therefore, there are $100 - 4 + 1 = 97$ leap years every 400 years. It follows that 303 years out of every 400 have 365 days and the rest have 366 days. Therefore the average number of days in a year is $\frac{365*303+366*97}{400} = \boxed{365\frac{97}{400}}$.

6. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are *arithmetically sequenced.*

Solution: For each *arithmetically sequenced* number, we can assign it a *diff-value* D where D is the common difference between consecutive digits of the number. From here we will do casework

on the value of the *diff-value* D:
Case 1: $D = 2$
Let the first digit be a. Then the next digits are $a+2, a+4$, and $a+6$. Therefore, we must have $a+6 \leq 9$ and $a \geq 1$. It follows that $a \leq 3$. Therefore we have 3 possibilities.
Case 2: $D = 1$
By similar logic, we have $a \geq 1$ and $a+3 \leq 9$. Therefore $a \leq 6$, and we have 6 possibilities.
Case 3: $D = 0$
By similar logic, we have $a \geq 1$ and $a \leq 9$. Therefore we have 9 possibilities.
Case 4: $D = -1$
By similar logic, we have $a \leq 9$ and $a - 3 \geq 0$. Therefore we have 7 possibilities.
Case 5: $D = -2$
By similar logic, we have $a \leq 9$ and $a - 6 \geq 0$. Therefore we have 4 possibilities.
Case 6: $D = -3$
By similar logic, we have $a \leq 9$ and $a - 9 \geq 0$. Therefore we have 1 possibility.
Adding up all of our cases, we have a total of $3+6+9+7+4+1 = \boxed{30}$ *arithmetically sequenced* 4-digit numbers.

7. Queen Jack likes a 5-card hand if and only if the hand contains only queens and jacks. Considering all possible 5-card hands that can come from a standard 52-card deck, how many hands does Queen Jack like?

Solution: Notice that in total there are $4 \times 2 = 8$ cards which are queens or jacks. We wish to choose 5 of these for a hand, so in total there are $\binom{8}{5} = \frac{8\times7\times6}{3\times2\times1} = \boxed{56}$ total hands.

Note: This solution used combinations. If you have not seen combinations before, please check this link: AoPS Combinations

8. $\mathbb{R}^2$-tic-tac-toe is a game where two players take turns putting red and blue points anywhere on the xy plane. The red player moves first. The first player to get 3 of their points in a line without any of their opponent's points in between wins. What is the least number of moves in which Red can guarantee a win? (We count each time that Red places a point as a move, including when Red places its winning point.)

Solution: Notice that it does not matter where the red player places its first points as all points in the plane are symmetric. For simplicity, place the first red point, R_1, at the origin. Notice that it does not matter where the blue player places their first point, as all points can be scaled and rotated to get any other point. For simplicity, place its first point, B_1, at $(1, 0)$. To maximize its chances of getting three in a row, Red will respond by placing a point which does not lie on the segment connecting $(0, 0)$ and $(1, 0)$. For simplicity, let it place its second point, R_2 at $(0, 1)$. Blue has to place a point between $(0, 0)$ and $(0, 1)$ to remain in the game. For simplicity, let it place its second point, B_2, at $(0, 0.5)$. Red has to place a point between $(1, 0)$ and $(0, 0.5)$ to remain in the game. For simplicity, let it place its third point, R_3, at $(0.5, 0.25)$. In response, Blue faces a dilemma. They have to prevent Red from winning by placing a point between both R_1 and R_3 and R_3 and R_2. This is impossible, so Blue can place its point wherever it wants, and then red will respond by guaranteeing a win on their $\boxed{4}$th move.

9. Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

Solution: Replace Alice and Bob with one combined person: AliceBob. Replace Claire and

Derek with one combined person: ClaireDerek. In each of these combined persons, one person in the pair stands next to the other person in the pair in one of 2 orders. Now we have 6 total independent "people", and there are $6! = 720$ ways to arrange them. However, each pair can rearrange themselves in 2 ways, so in total there are $2 \times 2 \times 720 = \boxed{2880}$ total different ways for the eight people to pose for their photo.

10. Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5 - x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart? (Note that $\lceil y \rceil$ denotes the smallest integer greater than or equal to y)

Solution: When computing an expected value over an area, we can multiply each area by its value, add up the results, and divide by the total area. Notice the area which is within 1 unit of the center is $1^2 \times \pi = \pi$. Each point in this area has a value of 5, so in total the value of this area is $5 \times \pi = 5\pi$. The area which is within 2 units of the center but not within 1 unit of the center is $(2^2 - 1^2) \times \pi = 3\pi$. Each point in this area has a value of 4, so in total the value of this area is $4 \times 3\pi = 12\pi$. In a similar fashion, the other areas have values of $15\pi, 14\pi$, and 9π. Adding these up, we get $(5 + 12 + 15 + 14 + 9)\pi = 55\pi$. Dividing this by the area of the circle $5^2 \times \pi = 25\pi$, we get $\boxed{\frac{11}{5}}$.

11. A 3×6 grid is filled with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$ according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) No number appears more than once in a given row. Let N be the number of ways to fill the grid and let k be the largest positive integer such that 2^k divides N. What is k?

Solution: Consider the first 3 columns. They must have all of the numbers from 1 through 9, so in total there are 9! ways to arrange the numbers in the first three columns. Now, we must arrange the numbers 1 through 9 in the last three columns such that no row has more than 1 of any number. For simplicity, assume the first three columns were arranged in the form:

1	2	3	-	-	-
4	5	6	-	-	-
7	8	9	-	-	-

Consider the first row of the last three columns. We must choose 3 of the numbers $4 - 9$ to fill this row. We can divide this into two cases:
Case 1: This row is made up of the elements of one of the other two rows ($\{4, 5, 6\}$ or $\{7, 8, 9\}$)
In this case, we have 2 ways to choose which row makes up the first row of the last three columns, and after that the other rows will be filled automatically. There are $3! = 6$ ways to arrange each row, so in total we have $2 \times 6 \times 6 \times 6 \times 9!$ possibilities.
Case 2: This row is made up of one element from one of the other two rows and two elements from the other ($\{4, 5, 6\}$ or $\{7, 8, 9\}$)
In this case, we have 2 ways to choose which row makes up one element of the first row of the last three columns, 3 ways to choose which element from that row goes into the first row of the last three columns, and 3 ways to choose which element from the third row does not go into the first row of the last three columns. After this, there are 3 ways to choose which element of the original first row either goes into or is left out of the 2nd row. As a result, there are $2 \times 3 \times 3 \times 3 \times 6 \times 6 \times 6 \times 9!$

possibilities in this case.
In total, we have $(2\times6\times6\times6)\times(1+3\times3\times3)\times9!$ possibilities. This is $432\times28\times9!$. The largest power of 2 which divides 432 is $16=2^4$. The largest power of 2 which divides 28 is $4=2^2$. The exponent of the largest power of 2 which divides 9! is $\lfloor\frac{9}{2}\rfloor+\lfloor\frac{9}{4}\rfloor+\lfloor\frac{9}{8}\rfloor=7$. Thus, the exponent of the largest power of 2 that divides the answer if $7+2+4=\boxed{13}$

12. There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?

Solution: Use a table like the following to organize your work:

	Good	Bad	Total
Claim Good			60
Claim Bad		30	
Total	50		100

Filling in the table, we get that there are $100-50=50$ total people who are bad at math. There are $100-60=40$ total people who claim to be bad at math. Therefore, the number of people who are good at math but refuse to admit it is $40-30=\boxed{10}$

13. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1,3,3,3,3,3\}$ and $\{1,3,1,3,1,3,1,3\}$)

Solution 1: Notice that when order does not matter, the ways to make 16 include 5 3's and 1 1, 4 3's and 4 1's, 3 3's and 7 1's, 2 3's and 10 1's, 1 3 and 13 1's, and 16 1's. Notice when there are a $3's$ and b $1's$, the total number of ways to arrange them is $\binom{a+b}{a}$. Therefore, our answer is $\binom{6}{1}+\binom{8}{4}+\binom{10}{3}+\binom{12}{2}+\binom{14}{1}+\binom{16}{0}=6+70+120+66+14+1=\boxed{277}$.

Solution 2: Let F_n be the number of ordered sequences of 1's and 3's which sum to n. We wish to compute F_{16}. Notice that for each ordered sequence which sums to $n-3$, we can add a 3 to it to make an ordered sequence which sums to n. In addition, for each ordered sequence which sums to $n-1$, we can add a 1 to it to make an ordered sequence which sums to n. Therefore:

$$F_n = F_{n-1} + F_{n-3}$$

With this equation we can form the following table by manually computing the first few values of F_n and then using the recursive equation above, we get the following sequence for F_n starting from F_0: $1,1,1,2,3,4,6,9,13,19,28,41,60,88,129,189,\boxed{277}$

14. How many positive numbers up to and including 2012 have no repeating digits?

Solution: We do casework based on the number of digits:
Case 1: There is one digit
In total, there are 9 1-digit numbers.
Case 2: There are two digits
In total, there are 9 ways to choose the first digit, and 9 ways to choose a remaining second digit. $9\times9=81$.
Case 3: There are three digits
In total, there are 9 ways to choose the first digit, 9 ways to choose a remaining second digit, and 8 ways to choose a remaining third digit. $9\times9\times8=648$.

Case 4: There are four digits and the leftmost digit is 1
In total, there is 1 way to choose the first digit, 9 ways to choose a remaining second digit, 8 ways to choose a remaining third digit, and 7 ways to choose a remaining fourth digit. $1 \times 9 \times 8 \times 7 = 504$.
Case 5: There are four digits and the leftmost digit is 2 Our numbers can either begin 200... or 201.... Only the second option does not have repeating digits. The only numbers in the second category are $2010, 2011$, and 2012, all of which have repeating digits. Therefore, there are 0 options for this case.
Adding up all of our cases, we have a total of $9 + 81 + 648 + 504 = \boxed{1242}$.

15. Call a nonnegative integer k sparse when all pairs of 1's in the binary representation of k are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than 2^{17}?

Solution: We will do casework based on the number of 1's in the binary representation of k:
Case 1: There are 0 1's
Clearly, there is only 1 way, as only the number 0 has 0 1's.
Case 2: There is 1 1.
We have 17 digit places where the 1 can go, so there are 17 numbers in this case.
Case 3: There are 2 1's
We wish to rearrange 13 0's, a block of the form 001, and 1 1 such that the 1 is to the left of the block of the form 001. There are $\frac{15!}{13!} = 14 \times 15 = 210$ ways to rearrange these objects, and by symmetry, half of them will have the 1 to the left of the 001, so we have a total of $\frac{210}{2} = 105$ numbers in this case.
Case 4: There are 3 1's
By similar logic, we wish to rearrange 10 0's, 2 blocks of the form 001, and 1 1 such that the 1 is to the left of both blocks of the form 001. There are $\frac{13!}{10! \times 2!} = \frac{11 \times 12 \times 13}{2}$ ways to rearrange these objects, and by symmetry, $\frac{1}{3}$ of these will have the 1 to the left of the blocks of the form 001, so we have a total of $\frac{11 \times 12 \times 13}{2 \times 3} = 286$ numbers in this case.
Case 5: There are 4 1's
By similar logic, we wish to arrange 7 0's, 3 blocks of the form 001, and 1 1 such that the 1 is to the left of all blocks of the form 001. There are $\frac{11!}{7! \times 3!} = 1320$ ways to rearrange these objects, and by symmetry, $\frac{1}{4}$ of these will have the 1 to the left of the blocks of the form 001, so we have a total of $\frac{1320}{4} = 330$ numbers in this case.
Case 6: There are 5 1's
By similar logic, we wish to arrange 4 0's, 4 blocks of the form 001, and 1 1 such that the 1 is to the left of all blocks of the form 001. There are $\frac{9!}{4! \times 4!} = 630$ ways to rearrange these objects, and by a similar argument to the above, it follows that there are a total of $\frac{630}{5} = 126$ numbers in this case.
Case 7: There are 6 1's
By similar logic, we wish to arrange 1 0, 5 blocks of the form 001, and 1 1 such that the 1 is to the left of all blocks of the form 001. There are $\frac{7!}{5!} = 42$ ways to rearrange these objects, and by a similar argument to the above $\frac{42}{6} = 7$ numbers in this case.
Adding up all of our cases, we get a total of $1 + 17 + 105 + 286 + 330 + 126 + 7 = \boxed{872}$ sparse numbers.

3 Sources

1. 2014 Stanford Math Tournament General Problem 3
2. 2014 Stanford Math Tournament General Problem 8
3. 2014 Stanford Math Tournament General Problem 11
4. 2014 Stanford Math Tournament General Problem 14
5. 2014 Stanford Math Tournament General Problem 16
6. 2014 Stanford Math Tournament General Problem 17
7. 2013 Stanford Math Tournament General Problem 3
8. 2013 Stanford Math Tournament General Problem 13
9. 2013 Stanford Math Tournament General Problem 16
10. 2013 Stanford Math Tournament General Problem 20
11. 2013 Stanford Math Tournament General Problem 25
12. 2012 Stanford Math Tournament General Problem 5
13. 2012 Stanford Math Tournament General Problem 9
14. 2012 Stanford Math Tournament General Problem 10
15. 2011 Stanford Math Tournament General Problem 8