Combinatorics Handout 1

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1 Problems

1. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

2. Ted has 2 red socks, 2 blue socks, and 2 green socks. He grabs three different socks at random. Compute the probability that they are all different colors.

3. Alice and Bob are playing a game in which Alice has a $\frac{1}{3}$ probability of winning, a $\frac{1}{2}$ probability of tying, and a $\frac{1}{6}$ probability of losing. Given that Alice and Bob played a game which did not end in a tie, compute the probability that Alice won.

4. Let a 5-digit number be termed a *valley* number if the digits (not necessarily distinct) in the number $\overline{abcde}$ satisfy $a > b > c$ and $c < d < e$. Compute the number of *valley* numbers that start with 3.

5. Including the leap years, compute the average number of days in a year. Express your answer as a mixed number. (Years that are evenly divisible by 4 are leap years. However years that are evenly divisible by 100 are not leap years, unless they are also evenly divisible by 400, in which case they are leap years.)

6. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are *arithmetically sequenced*.

7. Queen Jack likes a 5-card hand if and only if the hand contains only queens and jacks. Considering all possible 5-card hands that can come from a standard 52-card deck, how many hands does Queen Jack like?

8. $\mathbb{R}^2$-tic-tac-toe is a game where two players take turns putting red and blue points anywhere on the xy plane. The red player moves first. The first player to get 3 of their points in a line without any of their opponent's points in between wins. What is the least number of moves in which Red can guarantee a win? (We count each time that Red places a point as a move, including when Red places its winning point.)

9. Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

10. Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5 - x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart? (Note that $\lceil y \rceil$ denotes the smallest integer greater than or equal to y)

11. A 3×6 grid is filled with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$

according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) No number appears more than once in a given row. Let N be the number of ways to fill the grid and let k be the largest positive integer such that 2^k divides N. What is k?

12. There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?

13. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$)

14. How many positive numbers up to and including 2012 have no repeating digits?

15. Call a nonnegative integer k sparse when all pairs of 1's in the binary representation of k are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than 2^{17}?

2 Sources

1. 2014 Stanford Math Tournament General Problem 3
2. 2014 Stanford Math Tournament General Problem 8
3. 2014 Stanford Math Tournament General Problem 11
4. 2014 Stanford Math Tournament General Problem 14
5. 2014 Stanford Math Tournament General Problem 16
6. 2014 Stanford Math Tournament General Problem 17
7. 2013 Stanford Math Tournament General Problem 3
8. 2013 Stanford Math Tournament General Problem 13
9. 2013 Stanford Math Tournament General Problem 16
10. 2013 Stanford Math Tournament General Problem 20
11. 2013 Stanford Math Tournament General Problem 25
12. 2012 Stanford Math Tournament General Problem 5
13. 2012 Stanford Math Tournament General Problem 9
14. 2012 Stanford Math Tournament General Problem 10
15. 2011 Stanford Math Tournament General Problem 8