Combinatorics Handout #4 Answers and Solutions

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1 Answers

1. 11

2. $\frac{143}{4096}$

3. 196

4. 180

5. 719

6. 49

7. 184

8. $\frac{4}{7}$

9. $\frac{1}{2}$

10. $\frac{1}{9}$

11. 64

12. 764

13. 27

14. 7

15. $\frac{264}{4165}$

2 Solutions

1. Suppose we have 2013 piles of coins, with the i^{th} pile containing exactly i coins. We wish to remove the coins in a series of steps. In each step, we are allowed to take away coins from as many piles as we wish, but we have to take the same number of coins from each pile. We cannot take away more coins than a pile actually has. What is the minimum number of steps we have to take?

Solution: Let $f(n)$ be the minimum number of steps when there are n piles. Clearly $f(1) = 1$ and $f(2) = 2$. It is also clear that $f(n+1) \geq f(n)$ for all n, as otherwise the method for removing $n+1$ piles in $f(n+1)$ could be used to remove n piles in $f(n+1)$, meaning that $f(n)$ was not the minimum number of moves to remove n piles. We will show via induction that when $2^x \leq n < 2^{x+1}$, $f(n) = x+1$. Clearly the claim is true when $x = 0$. Now assume the claim is true when $x = y$. Consider some number of piles n such that $2^{y+1} \leq n < 2^{y+2}$. We will begin by showing that these piles can be removed in $y+2$ moves. First, remove 2^{y+1} stones from any pile with at least 2^{y+1} stones. Now, by our assumption, it is possible to remove all of the other piles which originally had less than 2^{y+1} stones in at most $y+1$ moves. From here, we are done, as we can remove all of the remaining piles during these $y+1$ moves by removing the same number of stones on a given move from the pile which originally had a stones as from the pile which originally had $2^{y+1}+a$ stones. It follows that $f(n) \leq y+2$ and $f(n) \geq y+1$. Now we must show that it is impossible to remove all n piles in $y+1$ moves. It suffices to show that it is impossible to remove the piles containing $1, 2, 3, 4, \cdots$ and 2^{y+1} stones in $y+1$ moves. If it were possible, then each value in the set $1, 2, 3, 4, \cdots, 2^{y+1}$ can be expressed as the sum of a subset of the set of the numbers of stones that were removed during each of the y moves. There are at most 2^{y+1} subsets, and one of these is the empty set, so it follows that our inductive claim was correct.

It follows by our claim that our answer is $f(2013) = \boxed{11}$.

2. A coin is flipped until there is a head followed by two tails. What is the probability that this will take exactly 12 flips?

Solution: Let f_n be the number of ways to flip a coin n times such that a head followed by two tails never occurs. We wish to calculate f_9. Let a_n be the number of ways to flip a coin n times with this property such that the last flip is a heads. Let b_n be the number of ways to flip a coin n times with this property such that the last two flips are a heads and a tails, respectively. Let c_n be the number of ways to flip a coin n times with this property such that the last flip is a tails and it is not preceded by a heads. It's clear that $f_n = a_n + b_n + c_n$. Clearly, $c_n = 1$ for all n as once a heads has appeared, it is impossible to have more than one tails in a row. We can also find that $a_n = c_{n-1} + b_{n-1} + a_{n-1} = a_{n-1} + b_{n-1} + 1$ as a head can be appended to every permutation from c_{n-1} and b_{n-1} to produce a permutation in a_n. Finally, we can find that $b_n = a_{n-1}$. It follows that $a_n = a_{n-1} + a_{n-2} + 1$ and $b_n = a_{n-1}$. We can easily calculate that $a_1 = 1$, $a_2 = 2$, $b_1 = 0$, $b_2 = 1$, and $c_1 = 1$. Through brute force, it follows that $a_8 = 54$, $a_9 = 88$, $b_9 = 54$, and $c_9 = 1$. Therefore, $f_9 = 88 + 54 + 1 = 143$. Therefore, our answer is $\frac{143}{2^{12}} = \boxed{\frac{143}{4096}}$ as desired.

3. A number is called boxy if the number of its factors is a perfect square. Find the largest boxy number less than 200.

Solution: We will do casework on the number of factors. If the number has 1 factor, it must be 1. If the number has 4 factors, it must be of the form pq or pq^3. A little brute force tells us that the maximum number of this form less than 200 is 194. If the number has 9 factors, it must be of the form p^2q^2 or p^8. We can find that the largest number of one of these forms less than 200 is 196. If the number has 16 factors, it must be of the form p^{15}, p^7q, p^3qr, or $pqrs$. The largest number of this form less than 200 is 168. The smallest number with 25 factors is $16 \cdot 81 > 200$, so we are done. Therefore our answer is $\boxed{196}$.

4. Alice, Bob, Clara, David, Eve, Fred, Greg, Harriet, and Isaac are on a committee. They need to split into three subcommittees of three people each. If not subcommittee can be all male or all female, how many ways are there to do this?

Solution: Notice that there are 4 females and 5 males. It follows that two of the subcommittees have 1 female and the other has 2 females. There are $\binom{4}{2} = 6$ ways to choose which two females are in the same subcommittee. From here, there are 5 ways to choose which male gets paired in that subcommittee, and $\binom{4}{2} = 6$ ways to divide the remaining boys. It follows that our answer is

$6 \cdot 5 \cdot 6 = \boxed{180}$.

Note: My answer is different from the official answer of 360. As an official solution has not been released, I am not sure why our answers are different.

5. How many integers between 0 and 999 are not divisible by $7, 11$, or 13?

Solution: There are $1+\lfloor \frac{999}{7} \rfloor = 1+142 = 143$ multiples of 7 in this range. There are $1+\lfloor \frac{999}{11} \rfloor = 91$ multiples of 11 in this range. There are $1 + \lfloor \frac{999}{13} \rfloor = 77$ multiples of 13 in this range. However, by adding these numbers, we are over counting multiples of more than one of these numbers. There are $1 + \lfloor \frac{999}{77} \rfloor = 13$ multiples of 77 in this range. There are $1 + \lfloor \frac{999}{91} \rfloor = 11$ multiples of 91 in this range. There are $1 + \lfloor \frac{999}{143} \rfloor = 7$ multiples of 143 in this range. However, subtracting these multiples will undercount multiples of 1001. There is 1 multiple of 1001 in this range. It follows that the total number of multiples in this range is $143 + 91 + 77 - 13 - 11 - 7 + 1 = 281$. It follows that our answer is $1000 - 281 = \boxed{719}$.

6. Compute the number of ways to make 50 cents using only pennies, nickels, dimes, and quarters.

Solution: This question is equivalent to finding the coefficient of x^{50} in the expansion of $(1 + x^1 + x^2 + \cdots + x^{50})(1 + x^5 + \cdots + x^{50})(1 + x^{10} + \cdots + x^{50})(1 + x^{25} + x^{50})$. Through casework on the contribution of larger factors or quarters, we can find that the coefficient of this term is $36 + 12 + 1 = \boxed{49}$.

7. Given that there are 168 primes with 3 digits or less, how many numbers between 1 and 1000 inclusive have a prime number of factors?

Solution: We will do casework on the number of factors. If the number has 2 factors, it must be prime. There are 168 primes in this range. If the number has 3 factors, it must be the square of a prime. $31^2 = 961$ is the largest square in this range, so we want the number of primes less than or equal to 31. There are 11 primes in this range. If the number has 5 factors, it must be of the form p^4. The largest fourth power in this range is $5^4 = 625$. Therefore there are 3 numbers of this form. If the number has 7 factors, it must be of the form p^6. There are 2 numbers of this form. If the number has 11 factors, it must be of the form p^{10}. There are no numbers of this form, so we are done. It follows that our answer is $168 + 11 + 3 + 2 = \boxed{184}$.

Once again, my answer differs with the official solution of 183. I have checked my answer with a Python program, so I am not sure why our answers are different.

8. Your wardrobe contains two red socks, two green socks, two blue socks, and two yellow socks. It is currently dark right now, but you decide to pair up the socks randomly. What is the probability that none of the pairs are of the same color?

Solution: Consider a blue sock. The probability that it is not paired correctly is $\frac{6}{7}$. Now the remaining blue sock must be paired with one of the socks in one of the remaining pairs. This occurs with probability $\frac{4}{5}$. The other sock in that pair must be paired with one of the socks in the remaining pair. This occurs with probability $\frac{2}{3}$. However, there is also the case where both blue socks are paired with the same color, which occurs with probability $\frac{6}{7} \cdot \frac{1}{5} \cdot \frac{2}{3} = \frac{4}{35}$ Therefore our answer is $\frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} + \frac{4}{35} = \boxed{\dfrac{4}{7}}$ as desired.

9. Fred and George are playing a game, in which Fred flips 2014 coins and George flips 2015 coins. Fred wins if he flips at least as many heads as George does, and George wins if he flips more heads than Fred does. Determine the probability that Fred wins.

Solution: Consider any outcome where Fred gets f heads and George gets g heads. Clearly the outcome where their flips are switched, or where every original head becomes a tails and vice versa, is a different outcome where Fred gets $2014 - f$ heads and George gets $2015 - g$ heads as it is impossible for g to be equal to $2015 - g$. Notice that if $f \geq g$, then $2014 - f < 2015 - g$ and if $f < g$, then $2014 - f \geq 2015 - g$. Therefore, there is a one to one correspondence between outcomes where Fred wins and outcomes where George wins, and it follows that the probability that Fred wins is $\boxed{\frac{1}{2}}$.

10. Albert and Kevin are playing a game. Kevin has a 10% chance of winning any given round in the match. If Kevin wins the first game, he wins the match. If not, he requests that the match be extended to a best of 3. If he wins the best of 3, he wins the match. If not, then he requests the match be extended to a best of 5, and so forth. What is the probability that Kevin eventually wins the match? (A best of $2n + 1$ match consists of a series of rounds. The first person to reach $n + 1$ winning games wins the match)

Solution: As an official solution has not been released for this question, I will provide a highly technical solution. I am guessing this was not the intended solution for this question. Let C_n be the number of different ways for $2n+1$ games to take place between two people so that the first person wins $n+1$ games, and for any value of x less than n, during the first $2x+1$ games, the first person has won less than $x+1$ games. We wish to calculate $\frac{1}{10}\sum_{n=0}^{\infty} C_n(\frac{9}{10})^n$. We can notice both that $C_0 = 1$ and that $C_n = \sum_{x=0}^{n-1} C_x C_{n-1-x}$. It follows that C_n is the nth Catalan number. Remembering that the Generating Function for the Catalan numbers is $C_0 + C_1 x + C_2 x^2 + \cdots = \frac{1-\sqrt{1-4x}}{2x}$, it follows that our answer is $\frac{1}{10} \cdot \frac{1-\sqrt{1-4\cdot\frac{9}{100}}}{\frac{18}{100}} = \boxed{\frac{1}{9}}$.

11. Find the number of 5-digit n such that every digit of n is either $0, 1, 3$, or 4, and n is divisible by 15.

Solution: Notice that n must end in a 0 and the sum of n's digits must be a multiple of 3. It follows that either n has 0 digits which are 1's or 4's, or it has 3 digits which are 1's or 4's. If all of n's digits are 0's or 3's, then there are $2^3 = 8$ possibilities. If n has three digits which are 1's or 4's, then either n has another digit which is a 0, or it has a digit which is a 3. If the fifth digit is a 0, then there are $3 + 9 + 9 + 3 = 24$ possibilities. If the fifth digit is a 3, then there are $4 + 12 + 12 + 4 = 32$ possibilities. It follows that our answer is $24 + 32 + 8 = \boxed{64}$.

12. Find the number of functions from the set $\{1, 2, ..., 8\}$ to itself such that $f(f(x)) = x$ for all $1 \leq x \leq 8$.

Solution: Call two numbers x and y a pair if $f(x) = y$ and $f(y) = x$, and $x \neq y$. It follows that f must have some number of pairs and all of the other elements of f of the form z must satisfy $f(z) = z$. If there are 4 pairs, then the total number of possibilities is $7 \cdot 5 \cdot 3 \cdot 1 = 105$. If there are 3 pairs, then the number of possibilities is $\binom{8}{2} \cdot 5 \cdot 3 \cdot 1 = 420$. If there are 2 pairs, then the number of possibilities is $\binom{8}{4} \cdot 3 \cdot 1 = 210$. If there is 1 pair, then the number of possibilities is $\binom{8}{6} \cdot 1 = 28$. If there are 0 pairs, then the number of possibilities is 1. It follows that our answer is $105 + 420 + 210 + 28 + 1 = \boxed{764}$.

13. Find the number of non-negative integer solutions (x, y, z) of the equation

$$xyz + xy + yz + zx + x + y + z = 2014.$$

Solution: Notice that this equation is equivalent to $(x+1)(y+1)(z+1) = 2015$. It follows that we must find the number of triples (a,b,c) such that $abc = 2015$. $2015 = 5 \cdot 13 \cdot 31$. There are 3 ways to choose which element has a factor of 5, 3 ways to choose which element has a factor of 13 and there are 3 ways to choose which element has a factor of 31. It follows that our answer is $3 \cdot 3 \cdot 3 = \boxed{27}$.

14. How many proper subsets of $\{1,2,3,4,5,6\}$ are there such that the sum of the elements in the subset is equal to twice a number in the subset?

Solution: We will do casework on the sum of the elements in the subset. If the sum is 2, then 1 must be an element, but this is impossible. If the sum is 4, then 2 must be an element, but this is also impossible. If the sum is 6, then 3 must be an element, and this only occurs when the subset is $\{1,2,3\}$. If the sum is 8, then 4 must be an element, and this only occurs when the subset is $\{1,3,4\}$. If the sum is 10, then 5 must be an element, and this only occurs in the subsets $\{1,4,5\}$ and $\{2,3,5\}$. If the sum is 12, then 6 must be an element, and this only occurs in the subsets $\{1,5,6\}, \{2,4,6\}$, and $\{1,2,3,6\}$. It follows that our answer is $\boxed{7}$.

15. Thomas, Olga, Ken, and Edward are playing the card game SAND. Each draws a card from a 52 card deck. What is the probability that each player gets a different rank and a different suit from the others?

Solution: The probability that Olga's card is in a different rank and suit from Thomas's card is $\frac{36}{51}$. The probability that Ken's card is in a different rank and suit from both Olga's card and Thomas's card is $\frac{22}{50}$. The probability that Edward's card is in a final suit and rank is $\frac{10}{49}$. Multiplying these gives us a final probability of $\frac{36 \cdot 22 \cdot 10}{51 \cdot 50 \cdot 49} = \frac{12 \cdot 22}{85 \cdot 49} = \boxed{\dfrac{264}{4165}}$.

3 Sources

1. Berkeley Math Tournament Discrete Spring 2013 Problem 3
2. Berkeley Math Tournament Discrete Spring 2013 Problem 6
3. Berkeley Math Tournament Individual Fall 2014 Problem 4
4. Berkeley Math Tournament Individual Fall 2014 Problem 11
5. Berkeley Math Tournament Team Fall 2014 Problem 3
6. Berkeley Math Tournament Team Fall 2014 Problem 4
7. Berkeley Math Tournament Team Fall 2014 Problem 12
8. Berkeley Math Tournament Team Fall 2014 Problem 19
9. Berkeley Math Tournament Individual Spring 2014 Problem 5
10. Berkeley Math Tournament Individual Spring 2014 Problem 15
11. Berkeley Math Tournament Discrete Spring 2014 Problem 2
12. Berkeley Math Tournament Individual Fall 2015 Problem 10
13. Berkeley Math Tournament Individual Fall 2015 Problem 17
14. Berkeley Math Tournament Team Fall 2015 Problem 6
15. Berkeley Math Tournament Individual Fall 2015 Problem 8