Combinatorics Handout #5 Answers and Solutions

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January 30, 2017

1 Answers

1. 324

2. 8

3. 512

4. 5280

5. 362880

6. 45

7. 169

8. $\frac{225}{2}$

9. 96

10. 192

11. $1009 \cdot 2015!$

12. 35

13. 9

14. 120

15. $\frac{1}{10}$

2 Solutions

1. The boba shop sells four different types of milk tea, and William likes to get tea each weekday. If William refuses to have the same type of tea on successive days, how many different combinations could he get, Monday through Friday?

Solution: There are 4 choices for the type of tea William gets on Monday, and for each day after that, William only has 3 choices for the type of milk he will get. Because each day's choice is independent beyond these conditions, our answer is the product of his choices for each day, or $4 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \boxed{324}$.

2. Suppose we list the decimal representations of the positive even numbers from left to right. Determine the 2015^{th} digit in the list.

Solution: There are $\frac{10-2}{2} = 4$ 1-digit even integers, $\frac{100-10}{2} = 45$ 2-digit even integers, $\frac{1000-100}{2} = 450$ 3-digit even integers, and $\frac{10000-1000}{2} = 4500$ 4-digit even integers. It follows that the even numbers less than 1000 have a total of $4 + 2 \cdot 45 + 3 \cdot 450 = 1444$ digits. Therefore, we wish to find the $2015 - 1444 = 571^{\text{th}}$ digit among the 4-digit even numbers. This means we wish to find the 3^{rd} digit of the $\frac{568}{4} + 1 = 143^{\text{rd}}$ 4-digit even number. This number will be $1000 + 142 \cdot 2 = 1284$, and therefore the 2015^{th} digit of the original list is $\boxed{8}$.

3. Find the number of ways to partition a set of 10 elements, $S = \{1, 2, 3, ..., 10\}$ into two parts; that is, the number of <u>unordered</u> pairs $\{P, Q\}$ such that $P \cup Q = S$ and $P \cap Q = \emptyset$.

Solution: Consider the problem when the pairs $\{P, Q\}$ are instead ordered pairs (P, Q). Clearly, for each element in S, we have 2 options for which set it is in, and therefore the total number of ordered pairs is $2^{10} = 1024$. Because $P \neq Q$, it follows that the number of unordered pairs is $\frac{1024}{2} = \boxed{512}$.

4. An integer is between 0 and 99999 (inclusive) is chosen, and the digits of its decimal representation are summed. What is the probability that the sum will be 19?

Solution: We will proceed by complementary counting with stars and bars. Let the value of each digit be represented by that digit number of stars. Then we wish to divide 19 stars among 5 bins (or decimal places), and therefore we need 4 bars. The number of ways to do this is $\binom{19+4}{4} = 8855$. However, this count includes the possibilities where one of the bins contains more than 9 stars. Because $19 < 10 \cdot 2$, we must have that if one of the bins contains more than 9 stars, exactly one of the bins contains more than 9 stars. If one of the bins has 10 stars, we have 5 ways to choose which bin and then $\binom{9+3}{3} = \binom{12}{3}$ ways to divide the other stars for a total of $5 \cdot \binom{12}{3}$ ways. If a bin has 11 stars, we have $5 \cdot \binom{11}{3}$ ways. Continuing this pattern, we have a total of $5 \cdot (\binom{12}{3} + \cdots + \binom{3}{3}) = 5 \cdot \binom{13}{4} = 3575$ ways which shouldn't be counted. It follows that our answer is $8855 - 3575 = \boxed{5280}$.

5. We have 10 boxes of different sizes, each one big enough to contain all the smaller boxes when put side by side. We may nest the boxes however we want (and how deeply we want), as long as we put smaller boxes in larger ones. At the end, all boxes should be directly or indirectly nested in the largest box. How many ways can we nest the boxes?

Solution: Let F_n be the number of ways to arrange the boxes when there are n original boxes. We claim $F_n = (n-1)!$. This is equivalent to proving a one-to-one correspondence between arrangements of the $n-1$ smallest boxes and arrangements of the n boxes. Consider an arrangement of the $n-1$ smallest boxes in a row. All of these boxes will be contained in the largest box by default. For the remaining $n-1$ boxes, if a box is to the right of a box that is larger than it, then we will say that the smaller box is contained in the larger box, and otherwise it is not. Clearly each rearrangement will produce a unique valid arrangement of the boxes as if one box is contained in a larger box which is contained in an even larger box, then they must also appear in that order in the row. In addition, every arrangement of the boxes within eachother corresponds to a unique row. It follows that $F_n = (n-1)!$, and therefore our answer is $9! = \boxed{362880}$.

6. Alice is planning a trip from the Bay Area to one of 5 possible destinations (each of which is serviced by only 1 airport) and wants to book two flights, one to her destination and one returning. There are 3 airports within the Bay Area from which she may leave and to which she may return. In how many ways may she plan her flight itinerary?

Solution: There are 3 ways to choose the airport she exits from, 3 ways to choose the airport she returns to, and 5 possible destinations, so our answer is $3 \cdot 3 \cdot 9 = \boxed{45}$.

7. Three balloon vendors each offer two types of balloons - one offers red & blue, one offers blue & yellow, and one offers yellow & red. I like each vendor the same, so I must buy 7 balloons from each. How many different triples (x, y, z) are there such that I could buy x blue, y yellow, and z red balloons?

Solution: Notice that it is impossible for any of x, y, z to be greater than 14. Any other triple x, y, z which satisfies $x + y + z = 21$ satisfies the property though. Convince yourself that this is true. It follows that we wish to divide 21 stars among 3 bins, so we need 2 bars. The total number of ways to do this is $\binom{23}{2} = 253$. However, we must subtract the triples where one of x, y, z is greater than 14. If one of x, y, z is 15, then there are $3 \cdot \binom{7}{1}$ ways. It follows that the total number of incorrect triples is $3 \cdot (\binom{7}{1} + \cdots + \binom{1}{1}) = 3 \cdot \binom{8}{2} = 84$. It follows that our answer is $253 - 84 = \boxed{169}$.

8. There are 30 cities in the empire of Euleria. Every week, Martingale City runs a very well-known lottery. 900 visitors decide to take a trip around the empire, visiting a different city each week in some random order. 3 of these cities are inhabited by mathematicians, who will talk to all visitors about the laws of statistics. A visitor with this knowledge has probability 0 of buying a lottery ticket, else they have probability 0.5 of buying one. What is the expected number of visitors who will play the Martingale Lottery?

Solution: The probability that the visitors visit Martingale City before visiting the mathematicians is $\frac{1}{4}$. Therefore, the expected number of visitors who will play the lottery is $\frac{1}{4} \cdot \frac{1}{2} \cdot 900 = \boxed{\frac{225}{2}}$.

9. A fair 6-sided die is repeatedly rolled until a $1, 4, 5$, or 6 is rolled. What is the expected value of the product of all the rolls?

Solution: Let E be the expected value in question. Notice that if the first roll is a 2, then the expected remaining product is $2E$. Notice that if the first roll is a 3, then the expected remaining product is $3E$. It follows that $E = \frac{2E}{6} + \frac{3E}{6} + 1 + 4 + 5 + 6$, and simplifying we find that the expected product is $E = \boxed{96}$.

10. How many ways are there to place the numbers $2, 3, ..., 10$ in a 3×3 grid, such that any two numbers that share an edge are mutually prime?

Solution: Notice that there are 5 even numbers in this set, so these numbers must occupy the four corner squares and the middle square in some order. Notice that if 6 or 10 were in the middle square, then they would be adjacent to their odd prime factor which is not allowed. It follows that we have 3 indistinguishable choices for the center square. Now we will do casework on where the 6 and the 10 are in relation to each other. If they occupy adjacent corners, then 7 must be in the square between them, and 5 must be in the other square adjacent to the 6. The rest of the squares can fill in any order. Ignoring the possibilities for the middle square, it follows that this case has $4 \cdot 2 \cdot 2 \cdot 2 = 32$ possibilities. If the 6 and the 10 are in opposite corners, then the 5 and the 7 must be adjacent to the 6, and the rest of the squares can fill in any order. This gives us a total of $4 \cdot 2 \cdot 2 \cdot 2 = 32$ possibilities for this case. It follows that our answer is $3 \cdot (32 + 32) = \boxed{192}$.

11. Consider the set $S = \{1, 2, ..., 2015\}$. How many ways are there to choose 2015 distinct (possibly empty and possibly full) subsets $X_1, X_2, ..., X_{2015}$ of S such that X_i is strictly contained in X_{i+1} for all $1 \leq i \leq 2014$?

Solution: Notice that each subset in X_i must have between 0 and 2015 elements, inclusive. It follows that one size of elements must be skipped between 0 and 2015. If no subset has 0 elements, then clearly there are 2015! ways to choose the subsets, as that is the number of ways to order the elements which are included. If one subset has 0 elements and one subset has 2015 elements, then that means at some step 2 elements must be added. There are $\binom{2015}{2}$ ways to choose which two elements are added, and 2013! ways to choose the order in which the other elements are added. Finally, there are 2014 ways to choose the step at which 2 elements are added for a total of $2015! \cdot 1007$ ways in this case. Finally, if no subset has 2015 elements, then there are 2015 ways to choose which element is excluded and there are 2014! ways to choose the order the other elements are added for a total of 2015! ways. It follows that the total number of ways to choose the subsets is $\boxed{1009 \cdot 2015!}$.

12. You decide to flip a coin some number of times, and record each of the results. You stop flipping the coin once you have recorded either 20 heads, or 16 tails. What is the maximum number of times that you could have flipped the coin?

Solution: Notice that you can get at most 19 heads and at most 15 tails before finishing the process. It follows that the maximum number of flips is $19 + 15 + 1 = \boxed{35}$.

13. How many 8-digit positive integers have the property that the digits are strictly increasing from left to right? For instance, 12356789 is an example of such a number, while 12337889 is not.

Solution: Notice that for any 8 distinct digits, there is exactly one way to arrange them such that they are in strictly increasing order. It follows that our answer is $\binom{9}{8} = \boxed{9}$.

14. How many different ways are there to arrange the letters $MILKTEA$ such that TEA is a contiguous substring?

For reference, the term "contiguous substring" means that the letters TEA appear in that order, all next to one another. For example, $MITEALK$ would be such a string, while $TMIELKA$ would not be.

Solution: Notice that there are 5 ways to choose the location of TEA in the string, and 24 ways to arrange the remaining letters for a total of $5 \cdot 24 = \boxed{120}$ different arrangements.

15. Suppose you roll two fair 20-sided dice. What is the probability that their sum is divisible by 10?

Solution: Notice that regardless of what is rolled on the first die, there are exactly two possibilities on the second die which will make the sum divisible by 10. It follows that our answer is $\frac{2}{20} = \boxed{\frac{1}{10}}$.

3 Sources

1. 2015 Berkeley Math Tournament Spring Individual Problem 1
2. 2015 Berkeley Math Tournament Spring Individual Problem 2
3. 2015 Berkeley Math Tournament Spring Individual Problem 5
4. 2015 Berkeley Math Tournament Spring Individual Problem 8

5. 2015 Berkeley Math Tournament Spring Individual Problem 10
6. 2015 Berkeley Math Tournament Spring Discrete Problem 1
7. 2015 Berkeley Math Tournament Spring Discrete Problem 5
8. 2015 Berkeley Math Tournament Spring Discrete Problem 6
9. 2015 Berkeley Math Tournament Spring Team Problem 1
10. 2015 Berkeley Math Tournament Spring Team Problem 3
11. 2015 Berkeley Math Tournament Spring Team Problem 6
12. 2016 Berkeley Math Tournament Fall Individual Problem 2
13. 2016 Berkeley Math Tournament Fall Individual Problem 8
14. 2016 Berkeley Math Tournament Fall Individual Problem 10
15. 2016 Berkeley Math Tournament Fall Individual Problem 11

Note: No official answer key was given for the 2015 Berkeley Math Tournament Spring Individual Round, so I do not know whether my solutions are correct for those questions. If there are any problems with my solutions for those questions, please submit a correct solution in one of the submission forms on the UHS Math club website.