Combinatorics Handout #8 Answers and Solutions

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1 Answers

1. $\frac{3}{5}$

2. 24

3. $\frac{1}{12}$

4. 39

5. 5

6. 144

7. 64

8. 728

9. 9

10. 2080

11. 48

12. 26

13. 825

14. 26

15. 5

2 Problems

1. Consider a round table with six identical chairs. If four students and two teachers randomly take seats, what is the probability that two teachers will not sit next to each other?

Solution: Fix the location of an arbitrary teacher. There are two seats which are next to this teacher, and there are 5 possible locations for the second teacher. It follows that our answer is $\frac{5-2}{5} = \boxed{\frac{3}{5}}$.

2. There are three different French books and two different Spanish books. How many ways are there to arrange the books in a row on a shelf with all books of the same language grouped together?

Solution: There are $2! = 2$ ways to decide which group of books are first, there are $3! = 6$ ways to order the French books, and there are $2! = 2$ ways to order the Spanish books. It follows that our answer is $2 \cdot 6 \cdot 2 = \boxed{24}$.

3. Three points are randomly located on a circle. What is the probability that the shortest distance between each point is less than or equal to the radius of the circle?

Solution: Fix the location of an arbitrary point. When a second point is at exactly a distance of the radius of the circle from the first point, we must have that the arc between the two points is a 60° arc of the circle. It follows that the probability that the second point is within a 60° arc of the first point is $\frac{2 \cdot 60}{360} = \frac{1}{3}$. On average in these cases, the second point will be 30° away from the first point. It follows that on average, the probability that the third point will be within a radial distance of each of the other two points is $30° + 2 \cdot (60° - 30°) = 90°$. Therefore, the probability that the third point satisfies the condition is $\frac{90}{360} = \frac{1}{4}$. It follows that our answer is $\frac{1}{3} \cdot \frac{1}{4} = \boxed{\frac{1}{12}}$.

4. How many two digit positive integers are multiples of 3 and/or 7?

Solution: Between 10 and 99, there are $\frac{99}{3} - \frac{12}{3} + 1 = 30$ multiples of 3. Between 10 and 99, there are $\frac{98}{7} - \frac{14}{7} + 1 = 13$ multiples of 7. Finally, between 10 and 99, there are $\frac{84}{21} - \frac{21}{21} + 1 = 4$ multiples of 21. It follows that our answer is $30 + 13 - 4 = \boxed{39}$ two-digit multiples of 3 and/or 7.

5. How many ordered pairs (x, y) of positive integers x and y satisfy the equation $3x + 5y = 80$?

Solution: Notice that because the right hand side is a multiple of 5, we must have that $3x$ is a multiple of 5. Therefore, x can only be equal to $5, 10, 15, 20,$ or 25. It follows that we have $\boxed{5}$ total solutions.

6. Find the number of 10-tuples $(a_1, a_2, ..., a_{10})$ such that $a_i \in \{1, 2, 3\}$ for $1 \leq i \leq 10$, $a_i < a_{i+1}$ if $i = 1, 3, 5, 7, 9$ and $a_i > a_{i+1}$ if $i = 2, 4, 6, 8$.

Solution: Let F_n be the number of n-tuples with these properties such that $a_n = 1$, let G_n be the number of n-tuples with these properties such that $a_n = 2$, and let H_n be the number of n-tuples with these properties such that $a_n = 3$. We can easily see that, in general, when n is even, $F_n = 0$, $G_n = F_{n-1}$, and $H_n = G_{n-1} + F_{n-1}$. We can easily see that, in general, when n is odd, $F_n = G_{n-1} + H_{n-1}$, $G_n = H_{n-1}$, and $H_n = 0$. We can calculate that $F_n = 1, G_n = 1$, and $H_n = 1$. With these recurrences, we can generate the following table of values:

n	F_n	G_n	H_n
1	1	1	1
2	0	1	2
3	3	2	0
4	0	3	5
5	8	5	0
6	0	8	13
7	21	13	0
8	0	21	34
9	55	34	0
10	0	55	89

It follows that our answer is $F_{10} + G_{10} + H_{10} = 0 + 55 + 89 = \boxed{144}$.

7. Find the number of 5-tuples of positive integers $(x_1, x_2, x_3, x_4, x_5)$ such that $x_1 = x_5, x_i \neq x_{i+1}$ for $i = 1, 2, 3, 4,$, and $x_i + x_{i+1} \leq 6$ for $i = 1, 2, 3, 4$.

Solution: We will do casework on the value of x_1. If $x_1 = 5$, then we must have $x_2 = x_4 = 1$ and x_3 can be $2, 3, 4$, or 5. It follows that we have 4 possibilities. If $x_1 = 4$, then we must have $x_2, x_4 \leq 2$. Regardless of the values of x_2, and x_4, x_3 can be 3, or 4, so we have at least $2^3 = 8$ solutions. If $x_2 = x_4 = 2$, then x_3 can be 1. If $x_2 = x_4 = 1$, then x_3 can be 2 or 5. It follows that we have $8 + 1 + 2 = 11$ solutions. If $x_1 = 3$, then we must have $x_2, x_4 \leq 2$. It follows that we have the same number of solutions as when $x_1 = 4$, so we have 11 solutions in this case. If $x_1 = 2$, then we must have $x_2, x_4 \leq 4$. If $x_3 = 3, 4$, or 5, then we must have $x_2 = x_4 = 1$ for 3 solutions. If $x_3 = 2$, then we have $3^2 = 9$ solutions. Finally, if $x_3 = 1$, then we have $2^2 = 4$ solutions. It follows that we have $3 + 9 + 4 = 16$ solutions in this case. When $x_1 = x_5 = 1$, we must have $x_2, x_4 \leq 5$. If $x_3 = 1$, then we have $4^2 = 16$ possibilities. If $x_3 = 2$, then we have $2^2 = 4$ solutions. If $x_3 = 3$, then we have $1^2 = 1$ solution. If $x_3 = 4$, then we have $1^2 = 1$ solution. It follows that we have $16+4+1+1 = 22$ solutions in this case. It follows that our answer is $4+11+11+16+22 = \boxed{64}$.

8. Find the number of functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ such that for $k = 1, 2, 3, 4$, $f(k + 1) \leq f(k) + 1$.

Solution: Let A_n be the number of choices for $f(1), f(2), \cdots$ and $f(n)$ such that $f(n) = 1$. Let B_n be defined similarly such that $f(n) = 2$. Let C_n be defined similarly such that $f(n) = 3$. Let D_n be defined similarly such that $f(n) = 4$. Finally, let E_n be defined similarly such that $f(n) = 5$. We know that $A_1 = B_1 = C_1 = D_1 = E_1 = 1$ and that $A_n = A_{n-1} + B_{n-1} + C_{n-1} + D_{n-1} + E_{n-1}$, $B_n = A_{n-1}+B_{n-1}+C_{n-1}+D_{n-1}+E_{n-1}$, $C_n = B_{n-1}+C_{n-1}+D_{n-1}+E_{n-1}$, $D_n = C_{n-1}+D_{n-1}+E_{n-1}$, and $E_n = D_{n-1} + E_{n-1}$. It follows that we can create the following table of values:

n	A_n	B_n	C_n	D_n	E_n
1	1	1	1	1	1
2	5	5	4	3	2
3	19	19	14	9	5
4	66	66	47	28	14
5	221	221	155	89	42

It follows that our answer is $A_5 + B_5 + C_5 + D_5 + E_5 = 221 + 221 + 155 + 89 + 42 = \boxed{728}$.

9. Let $q(n)$ be the number of ways to express n as a sum of two positive integers, using each of them at least once. For example, since $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1$, we have $q(5) = 5$. Find the number of positive integers n such that $n \leq 100$, $n \equiv 3 \pmod 4$, and $q(n) \equiv 0 \pmod 2$.

Solution: Through brute force, we can find that $q(3) = 1$, $q(7) = 11$, $q(11) = 27$, $q(15) = 44$, $q(19) = 71$, $q(23) = 97$, $q(27) = 126$, $q(31) = 157$, $q(35) = 182$, $q(39) = 230$, $q(43) = 259$, $q(47) = 295$, $q(51) = 352$, $q(55) = 368$, $q(59) = 413$, $q(63) = 475$, $q(67) = 499$, $q(71) = 541$, $q(75) = 629$, $q(79) = 631$, $q(83) = 677$, $q(87) = 784$, $q(91) = 764$, $q(95) = 824$, and $q(99) = 935$. It follows that $q(4x + 3)$ is even when $x = 3, 6, 8, 9, 12, 13, 21, 22$, and 23. Therefore, there are $\boxed{9}$ positive integers n which work.

10. In a 2×6 matrix, we want to fill 1 or 2 in each term. Also, for $i = 1, 2, 3, 4, 5, 6$, define c_i as the product of the terms in the ith column. How many ways are there to fill the terms so that

$$\sum_{i=1}^{6} c_i \equiv 0 \pmod 2?$$

Solution: We will do casework on the number of columns with odd products. If all 6 columns have odd products, then there is 1 possibility. If 4 columns have odd products, then there are $\binom{6}{2} \cdot 3^2 = 135$ possibilities. If 2 columns have odd products, then there are $\binom{6}{2} \cdot 3^4 = 1215$ possibilities. If 0 columns have odd products, then there are $3^6 = 729$ possibilities. It follows that our answer is $1 + 135 + 1215 + 729 = \boxed{2080}$.

11. In the set $\{1, 2, 3, ..., 8\}$, how many subsets contain 4 consecutive integers?

Solution: We will do casework on the length of the longest run of consecutive integers in our subset. If the subset contains 8 consecutive integers, then there is only 1 subset. If the subset contains 7 consecutive integers, then there are 2 subsets. If the subset contains 6 consecutive integers, then there are $2 \cdot 2 + 1 = 5$ subsets. If the subset contains 5 consecutive integers, then there are $2 \cdot 2^2 + 2 \cdot 2^1 = 12$ subsets. Finally, if the subset contains 4 consecutive integers, then there are $2 \cdot 2^3 + 3 \cdot 2^2 = 28$ subsets. It follows that our answer is $1 + 2 + 5 + 12 + 28 = \boxed{48}$.

12. In a regular 20-gon with 1 as the length of all sides, pick 5 points to make a pentagon. How many pentagons have all of its sides larger than 2? If two pentagons are the same when rotated, they are still considered to be different.

Solution: We can easily find that the radius of the circumcircle of the regular 20-gon is $\frac{1}{2\sin(9^\circ)}$. It follows that if a segment connects the ends of a minor arc which contains x edges of the 20-gon, then the length of that segment is $\frac{\sin(9x^\circ)}{\sin 9^\circ}$. It follows that we wish to calculate the minimum value of x such that $\sin(9x^\circ) > 2\sin(9^\circ)$. By $\sin(2x) = 2\sin(x)\cos(x)$, we know that $\sin(18^\circ) < 2\sin(9^\circ)$. By $\sin(3x) = 3\sin(x) - 4\sin^3(x)$, we know that $\frac{\sin(27^\circ)}{\sin(9^\circ)} = 3 - 4\sin^2(9^\circ)$. Because $4\sin^2(x) = 1$ when $x = 30^\circ$, we know that $\sin(27\circ) > 2\sin(9^\circ)$. It follows that any pentagon where all sides pass over more than 2 sides of the 20-gon will work. It follows that we wish to investigate sums of 5 numbers which are all greater than or equal to 3 which evaluate to 20. We can find that the only sums that work are $3+3+3+3+8, 3+3+3+4+7, 3+3+3+5+6, 3+3+4+4+6, 3+3+4+5+5, 3+4+4+4+5$, and $4+4+4+4+4$. The first sum results in exactly 1 pentagon. The second sum results in exactly 4 pentagons. The third sum results in 4 pentagons. The fourth sum results in 6 pentagons. The fifth sum results in 6 pentagons. The sixth sum results in 4 pentagons. The seventh sum results in 1 pentagon. It follows that our answer is $1 + 4 + 4 + 6 + 6 + 4 + 1 = \boxed{26}$.

13. We want to choose 8 people out of 20 people who are sitting in a circle. We do not want to choose two people who are next to each other. Calculate how many ways are possible.

Solution: There are 20 ways to choose an arbitrary person as our "leftmost" person. From here, we wish to choose positive integers a, b, c, d, e, f, g, and h such that $a+1+b+1+c+1+d+1+e+1+f+1+g+1+h = 19$, or $a+b+c+d+e+f+g+h = 12$. By Stars and Bars, it follows that there are $\begin{pmatrix} 11 \\ 7 \end{pmatrix}$ ways to do this. It follows that in total, there are $20 \cdot \begin{pmatrix} 11 \\ 7 \end{pmatrix}$ possibilities. However, because the "leftmost" person could be any of the 8 people in our set, we must divide by 8. It follows that our answer is $\frac{330 \cdot 20}{8} = \boxed{825}$.

14. How many 5-digit numbers are there such that all digits are either $1, 2, 3$, or 4 and no two digits next to each other differ by 1?

Solution: Let f_n be the number of n-digit numbers with this property such that the units digit is either a 1 or a 4. Let g_n be the number of n-digit numbers with this property such that the units digit is either a 2 or a 3. Then we have that $f_1 = g_1 = 2$, $g_n = f_{n-1}$ and $f_n = g_{n-1} + f_{n-1}$. Using these properties, we can create the following table:

n	f_n	g_n
1	2	2
2	4	2
3	6	4
4	10	6
5	16	10

It follows that our answer is $f_5 + g_5 = 16 + 10 = \boxed{26}$.

15. Find the number of subsets of $\{1, 2, ..., 23\}$ such that the number of elements is 11 and the sum of the elements is 194.

Solution: Notice that the maximum sum of 11 elements is $23 \cdot 11 - (1+2+\cdots+10) = 253 - 55 = 198$. It follows that there is a 1 to 1 correspondence between subsets with a sum of 194 and subsets with a sum of $1 + 2 + \cdots + 11 + 4 = 70$. Notice that it is impossible to increase an element between 1 and 7 without having 2 of the same elements in the subset. Therefore, we only need to increase the elements of $\{8, 9, 10, 11\}$ by a collective total of 4. Through brute force, we can find that the only options are $\{9, 10, 11, 12\}$, $\{8, 10, 11, 13\}$, $\{8, 9, 11, 14\}$, $\{8, 9, 12, 13\}$, and $\{8, 9, 10, 15\}$. It follows that our answer is $\boxed{5}$.

3 Sources

1. KSEA National Mathematics Competition 2007 11th Grade Problem 2 (Korea)
2. KSEA National Mathematics Competition 2007 10th Grade Problem 11 (Korea)
3. KSEA National Mathematics Competition 2007 10th Grade Problem 15 (Korea)
4. KSEA National Mathematics Competition 2007 9th Grade Problem 7 (Korea)
5. KSEA National Mathematics Competition 2007 9th Grade Problem 10 (Korea)
6. Korean Mathematical Olympiad First Round 2015 Problem 2
7. Korean Mathematical Olympiad First Round 2015 Problem 7
8. Korean Mathematical Olympiad First Round 2015 Problem 10
9. Korean Mathematical Olympiad First Round 2015 Problem 12
10. Korean Mathematical Olympiad First Round 2014 Problem 2
11. Korean Mathematical Olympiad First Round 2014 Problem 5
12. Korean Mathematical Olympiad First Round 2014 Problem 10
13. Korean Mathematical Olympiad First Round 2013 Problem 1
14. Korean Mathematical Olympiad First Round 2013 Problem 12
15. Korean Mathematical Olympiad First Round 2012 Problem 1 (Adapted)