Algebra Handout 2 Answers and Solutions

Walker Kroubalkian

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1 Answers

1. 8

2. 35

3. 64π

4. -71

5. 33

6. 200

7. $\frac{5}{7}$

8. 50

9. 340

10. 3

11. 144169

12. $\frac{9}{1-2f(x)+f(x)^2}$ or $f(f(x))^2$

13. -21

14. $50, 47$

15. $-\frac{3}{11}$

2 Solutions

1. You have a bag containing 3 types of pens: red, green, and blue. 30% of the pens are red pens, and 20% are green pens. If, after you add 10 blue pens, 60% of the pens are blue pens, how many green pens did you start with?

Solution: Let the original number of pens be $2b$. Because $20\% + 30\% = 50\%$, we know that the number of blue pens is b. Therefore, we wish to solve $\frac{b+10}{2b+10} = 60\% = \frac{3}{5}$. Solving, we get $b = 20$, so the initial number of pens was 40. Therefore our answer is $20\% \cdot 40 = \boxed{8}$.

2. Bacon, Meat, and Tomato are dealing with paperwork. Bacon can fill out 5 forms in 3 minutes, Meat can fill out 7 forms in 5 minutes, and Tomato can staple 3 forms in 1 minute. A form must be filled out and stapled together (in either order) to complete it. How long will it take them to complete 105 forms?

Solution: Together, Bacon and Meat can fill out $5 \cdot 5 + 7 \cdot 3 = 46$ forms in 15 minutes. Therefore, it will take them $15 \cdot \frac{105}{46} = \frac{1575}{46}$ minutes to fill out 105 forms. It will take Tomato $\frac{105}{3} = 35$ minutes to staple 105 forms. Because $\frac{1575}{46} < 35$, we have that it will take them $\boxed{35}$ minutes to complete 105 forms.

3. What is the volume of a cylinder whose radius is equal to its height and whose surface area is numerically equal to its volume?

Solution: Let the radius of the cylinder be r. We know the volume of the cylinder is πr^3, and we know the surface area of the cylinder is $2\pi r^2 + 2\pi r^2 = 4\pi r^2$. Therefore, we wish to solve $\pi r^3 = 4\pi r^2$, which gives us $r = 4$. From here, we have that the volume of the cylinder is $4^3 \cdot \pi = \boxed{64\pi}$.

4. Given for some real a, b, c, d,

$$P(x) = ax^4 + bx^3 + cx^2 + dx$$

$$P(-5) = P(-2) = P(2) = P(5) = 1$$

Find P(10).

Solution: Consider the polynomial $G(x) = P(x) - 1$. We know $P(x)$ is a quartic, so we know $G(x)$ is a quartic with roots of -5, -2, 2, and 5. It follows that $G(x) = a(x+5)(x+2)(x-2)(x-5)$. Therefore, $P(x) = a(x+5)(x+2)(x-2)(x-5)+1$. Notice that the constant term of $G(x)$ is $100a$. If $100a + 1 = 0$, we must have $a = -\frac{1}{100}$. It follows that $P(10) = -\frac{1}{100}(10+5)(10+2)(10-2)(10-5)+1 = \boxed{-71}$.

5. Allison wants to bake a pie, so she goes to the discount market with a handful of dollar bills. The discount market sells different fruit each for some integer number of cents and does not add tax to purchases. She buys 22 apples and 7 boxes of blueberries, using all of her dollar bills. When she arrives back home, she realizes she needs more fruit, though, so she grabs her checkbook and heads back to the market. This time, she buys 31 apples and 4 boxes of blueberries, for a total of 60 cents more than her last visit. Given she spent less than 100 dollars over the two trips, how much (in dollars) did she spend on her first trip to the market?

Solution: Let the price of an apple be a dollars, and let the price of a blueberry be b dollars. Let the number of dollars Allison spent on the first trip be x. Then we have $22a + 7b = x$, and $31a + 4b = x + 0.6$. Subtracting the first equation from the second equation gives us $9a - 3b = 0.6$, or $3a - b = 0.2$. Solving this for b gives us $b = 3a - 0.2$. If we multiply the first equation by -4 and the second equation by 7, and then add the results, we get $129a = 3x + 4.2$ or $43a - 1.4 = x$. If we let the price of an apple be c cents, then it follows that $0.43c - 1.4 = x$ or $43c - 140 = 100x$. It follows that $43c \equiv 40 \pmod{100}$. From here, we can determine that $c \equiv 80 \pmod{100}$, so we know that the price of an apple is some number of dollars plus 80 cents. Notice that the total amount she spent over the two trips is $2x + 0.6 = 86a - 2.2$. If a were 1.8, then we would have $86a - 2.2 > 100$, so we must have $a = 0.8$. It follows that $b = 3 \cdot 0.8 - 0.2 = 2.2$, and $22a + 7b = x = 22 \cdot 0.8 + 7 \cdot 2.2 = \boxed{33}$ as desired.

6. Ed, Fred, and George are playing on a see-saw that is slightly off center. When Ed sits on the left side and George, who weighs 100 pounds, sits on the right side, they are perfectly balanced. Similarly, if Fred, who weighs 400 pounds, sits on the left and Ed sits on the right, they are also perfectly balanced. Assuming the see-saw has negligible weight, what is the weight of Ed, in pounds?

Solution: Let the length of the left side be l and the length of the right side be r. By the Law of the Lever, if a pounds are placed on the left side and b pounds are placed on the right side, the lever will only be balanced if $l \cdot a = b \cdot r$. It follows that if Ed weighs E pounds, George weighs 100 pounds, and Fred weighs 400 pounds, we must have $l \cdot E = 100r$ and $400l = r \cdot E$. Adding these equations we get $l(400+E) = r(100+E)$ or $\frac{l}{r} = \frac{100+E}{400+E}$. From the first equation, we know $\frac{lE}{r} = 100$, so we know that $\frac{100E+E^2}{400+E} = 100$. Simplifying, we get $E^2 = 40000$, or $E = \boxed{200}$ as desired.

7. Let $f(x) = 6x + 7$ and $g(x) = 7x + 6$. Find the value of a such that $g^{-1}(f^{-1}(g(f(a)))) = 1$.

Solution: We will proceed by working backwards. If $g^{-1}(y) = 1$, then we must have $g(1) = y$, or $y = 7 \cdot 1 + 6 = 13$. If $f^{-1}(z) = 13$, then we must have $f(13) = z$, or $z = 6 \cdot 13 + 7 = 85$. If $g(w) = 85$, then we must have $7w + 6 = 85$, or $w = \frac{79}{7}$. If $f(a) = \frac{79}{7}$, then we must have $6a + 7 = \frac{79}{7}$, or $a = \boxed{\frac{5}{7}}$.

8. Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic progression with $a_1 > 0$ and $5a_{13} = 6a_{19}$. What is the smallest integer n such that $a_n < 0$?

Solution: Let the initial term be $a_1 = x$ and the common difference be d. Then we have $5(x + 12d) = 6(x+18d)$, or $x = -48d$. Because $x > 0$, we must have $d < 0$. Now, because $a_n = x+(n-1)d$, we must have $a_n = -48d + (n-1)d < 0$. Clearly, the smallest value of n that satisfies this is $49 + 1 = \boxed{50}$.

9. Simplify

$$[\log_{xyz}(x^z)][1 + \log_x y + \log_x z]$$

where $x = 628, y = 233, z = 340$.

Solution: Notice that $1 + \log_x y + \log_x z = \log_x x + \log_x y + \log_x z = \log_x xyz$. Therefore, we wish to compute $[\log_x xyz][\log_{xyz} x^z]$. By the properties of logarithms, this is equivalent to $\log_x x^z = z = \boxed{340}$.

10. The roots of the equation $x^3 + ax^2 + bx + c = 0$ are three consecutive integers. Find the maximum value of $\frac{a^2}{b+1}$.

Solution: Let the roots be $r-1$, r, and $r+1$. By Vieta's Equations, we know $r - 1 + r + r + 1 = 3r = -a$. We also know that $b = (r-1)r + (r+1)r + (r-1)(r+1) = 3r^2 - 1$. Therefore, we wish to maximize $\frac{(-3r)^2}{3r^2-1+1} = 3$ when r is nonzero. Therefore, our answer is $\boxed{3}$.

11. Denote $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. What is $144169 * S_{144169} - (S_1 + S_2 + \cdots + S_{144168})$?

Solution: Notice that $S_1 + S_2 + S_3 + \cdots + S_n = \sum_{x=1}^{n} \sum_{i=1}^{x} \frac{1}{i} = \sum_{x=1}^{n} \frac{n+1-x}{x} = \sum_{x=1}^{n} (\frac{n+1}{x} - 1)$. Therefore, $S_1+S_2+S_3+\cdots+S_{144168} = \sum_{x=1}^{144168} \frac{144169}{x} - 144168$. We also know that $144169S_{144169} = \sum_{x=1}^{144169} \frac{144169}{x}$. Therefore, our sum is $\sum_{x=1}^{144169} \frac{144169}{x} - (\sum_{x=1}^{144168} (\frac{144169}{x}) - 144168) = \frac{144169}{144169} + 144168 = \boxed{144169}$.

12. Given $f(x) = \frac{3}{x-1}$, then express $\frac{9(x^2-2x+1)}{x^2-8x+16}$ entirely in terms of $f(x)$. In other words, x should not be in your answer, only $f(x)$.

Solution: We wish to express $(3 \cdot \frac{x-1}{x-4})^2$ in terms of $\frac{3}{x-1}$. Notice that $1 - f(x) = \frac{x-1}{x-1} - \frac{3}{x-1} = \frac{x-4}{x-1}$, so our expression is $(\frac{3}{1-f(x)})^2 = \boxed{\frac{9}{1 - 2f(x) + f(x)^2}}$

Note: This is also equivalent to $f(f(x))^2$.

13. Suppose that $f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, and that $f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = 7$. What is a?

Solution: Consider the function $g(x) = f(x) - 7$. We know that $g(x)$ is a 6th degree monic polynomial with zeroes at $1, 2, 3, 4, 5$, and 6. Therefore, $g(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$ and $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) + 7$. From here, it is easy to see that the coefficient of x^5 in $f(x)$ is $-1-2-3-4-5-6 = \boxed{-21}$.

14. There are two non-consecutive positive integers a, b such that $a^2 - b^2 = 291$. Find a and b.

Solution: Notice that this equation is equivalent to $(a-b)(a+b) = 291 = 3 \cdot 97$. Because a and b are non-consecutive, we know $a - b$ is not 1. Because a and b are positive, we know $a - b < a + b$. It follows $a - b = 3$ and $a + b = 97$. Solving, we get $a = 50$ and $b = 47$. Therefore, our answer is $\boxed{50, 47}$.

15. Let r_1, r_2, r_3 be the roots of $f(x) = x^3 - 6x^2 - 5x + 22$. Find $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$.

Solution: Notice that $r_1 r_2 = \frac{r_1 r_2 r_3}{r_3}$. By Vieta's Equations, we know $r_1 r_2 r_3 = -22$. Therefore, we wish to find $\frac{1}{\frac{-22}{r_3}} + \frac{1}{\frac{-22}{r_2}} + \frac{1}{\frac{-22}{r_1}} = -\frac{1}{22}(r_1 + r_2 + r_3)$. By Vieta's Equations, we also know $r_1 + r_2 + r_3 = 6$. Therefore, our answer is $\frac{-6}{22} = \boxed{-\frac{3}{11}}$.

3 Sources

1. 2012 Berkeley Math Tournament Fall Individual Problem 5
2. 2012 Berkeley Math Tournament Fall Individual Problem 7
3. 2012 Berkeley Math Tournament Fall Individual Problem 12
4. 2012 Berkeley Math Tournament Fall Individual Problem 20
5. 2012 Berkeley Math Tournament Fall Individual Problem 18
6. 2012 Berkeley Math Tournament Fall Team Problem 1
7. 2012 Berkeley Math Tournament Fall Team Problem 6
8. 2012 Berkeley Math Tournament Spring Individual Problem 1
9. 2012 Berkeley Math Tournament Spring Tournament Round 1 Problem 2
10. 2012 Berkeley Math Tournament Spring Tournament Round 3 Problem 4
11. 2012 Berkeley Math Tournament Spring Tournament Round 4 Problem 1
12. 2012 Berkeley Math Tournament Spring Tournament Round 5 Problem 3
13. 2012 Berkeley Math Tournament Spring Tournament Championship Round Problem 3
14. 2013 Berkeley Math Tournament Fall Team Problem 10
15. 2013 Berkeley Math Tournament Fall Speed Problem 89