Algebra Handout 3 Answers and Solutions

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November 28, 2017

1 Answers

1. $(2, 1)$

2. -3

3. 19

4. $\frac{4}{7}$

5. $\frac{3+\sqrt{26}}{2}$

6. 439

7. $\frac{2\sqrt{11}+\sqrt{41}+3}{4}$

8. $(3, \frac{1}{3}), (1, 3)$

9. $\frac{257}{16}$

10. 2

11. $\frac{3}{2}$

12. -34

13. 1

14. -3193

15. 23

2 Solutions

1. Find all ordered pairs (x, y) such that $(x - 2y)^2 + (y - 1)^2 = 0$.

Solution: Notice that because the Left Hand Side (LHS) of this equation is the sum of perfect squares, the LHS can only be equal to 0 if each perfect square is equal to 0. It follows that $y-1 = 0$ and $x - 2y = 0$. From here, it is easy to see that $y = 1$ and $x = 2$. It follows that the only solution is $\boxed{(2, 1)}$ as desired.

2. Find the product of all real x for which $2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3} = 0$.

Solution: Let $2^x = a$. It follows that $2a^3 - 17a^2 + 8a = 0$. Because $2^x \neq 0$, it follows that

$2a^2 - 17a + 8 = 0$. Using the Quadratic Formula, it follows that $a = \frac{1}{2}$ or $a = 8$. It follows that $x = -1$ or $x = 3$, and therefore our answer is $\boxed{-3}$.

3. Find the largest positive integer n such that $n^3 + 4n^2 - 15n - 18$ is the cube of an integer.

Solution: Notice that when n is large, $n^3 + 4n^2 - 15n - 18 > n^3$ and $n^3 + 4n^2 - 15n - 18 < n^3 + 6n^2 + 12n + 8 = (n+2)^3$. It follows that $n^3 + 4n^2 - 15n - 18 = (n+1)^3 = n^3 + 3n^2 + 3n + 1$, and from here it follows that $n^2 - 18n - 19 = (n-19)(n+1) = 0$. It follows that the largest integer n which satisfies this property is $n = \boxed{19}$.

4. Given that $a + b + c = 5$ and that $1 \leq a, b, c \leq 2$, what is the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$?

Solution: Notice that we wish to minimize $\frac{1}{a+b} + \frac{1}{b+c} = \frac{1}{5-c} + \frac{1}{5-a}$. This will clearly be minimized when a and c are minimized, so it makes sense for b to be equal to 2, making $a + c = 3$. Now we wish to minimize $\frac{10-a-c}{(5-a)(5-c)} = \frac{7}{(5-a)(5-c)}$. Because $5 - a + 5 - c = 10 - 3 = 7$, by $AM - GM$, $\frac{7}{2} \geq \sqrt{(5-a)(5-c)}$. It follows that our minimum value is $\frac{7}{(\frac{7}{2})^2} = \boxed{\frac{4}{7}}$.

5. Find the maximum value of $x + y$, given that $x^2 + y^2 - 3y - 1 = 0$.

Solution: Notice that this equation is equivalent to $x^2 + (y - \frac{3}{2})^2 = \frac{13}{4}$ which is a circle of radius $\frac{\sqrt{13}}{2}$ centered at $(0, \frac{3}{2})$. The value of $x + y$ will be maximized at a point on this circle where the tangent line has a slope of -1. This will occur when the x-displacement and the y-displacement from the center of the circle to the point are the same, which occurs at a $45°$ angle with the positive x-direction. It follows that our answer is $x + y = \frac{3}{2} + \frac{\sqrt{26}}{4} + \frac{\sqrt{26}}{4} = \boxed{\frac{3+\sqrt{26}}{2}}$.

6. A polynomial P is of the form $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$. Given that $P(2) = 27$, what is $P(3)$?

Solution: Notice that when $x = 2$,

$$P(x) = \pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1 = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 - (\sum x^{n+1})$$

where x^n is a term in the expansion of $P(x)$ with a coefficient of -1. When $x = 2$, this can be rewritten as

$$P(x) = 127 - (\sum x^{n+1})$$

. Because n ranges from 0 to 6, the value of $\sum x^{n+1}$ can be any even integer between 0 and $2 \cdot 127 = 254$ inclusive. Because there are only $2^7 = 128$ possibilities for $P(x)$, and there are only 128 even integers in this range, it follows that there is exactly one polynomial $P(x)$ of this form such that $P(2) = 27$. By guessing and checking, we can find that $P(x) = x^6 - x^5 - x^4 + x^3 + x^2 - x + 1$ is the only polynomial P such that $P(2) = 27$. It follows that $P(3) = 729 - 243 - 81 + 27 + 9 - 3 + 1 = \boxed{439}$ as desired.

7. What is the sum of the positive solutions to $2x^2 - x \cdot \lfloor x \rfloor = 5$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x?

Solution: We will use the fact that $x = \lfloor x \rfloor + \{x\}$ where $\{x\}$ is the fractional part of x. Letting $\{x\} = a$ and $\lfloor x \rfloor = b$, we have that $2(a+b)^2 - (a+b)b = 5$. Rearranging this, we get $2a^2 + 3ab + b^2$. Because $a + b$ is positive, we must have $0 \leq b \leq 2$. When $b = 0$, we have that $2a^2 = 5$, resulting in $a^2 = \frac{5}{2}$, which is impossible. When $b = 1$, we have that $a = \frac{\sqrt{41}-3}{4}$ which is possible. When

$b = 2$, we have that $a = \frac{\sqrt{11}-3}{2}$ which is possible. Therefore, our answer is $1 + 2 + \frac{\sqrt{41}-3}{4} + \frac{\sqrt{11}-3}{2} = \boxed{\dfrac{2\sqrt{11} + \sqrt{41} + 3}{4}}$ as desired.

8. Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

Solution: Let $a = x$, and let $b = xy$. Then we have that $a + b = 4$ and $ab = 3$. By Vieta's equations, it follows that a and b are roots of $z^2 - 4z + 3 = 0$. Therefore, either $x = 3$ and $xy = 1$, or $x = 1$ and $xy = 3$. It follows that our solutions are $\boxed{(3, \frac{1}{3}), (1, 3)}$.

9. Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}.$$

Solution: Rationalizing the denominators of these fractions, we find that $\frac{\sqrt{x}-\sqrt{x-2}}{2} + \frac{\sqrt{x+2}-\sqrt{x}}{2} = \frac{1}{4}$. It follows that $\sqrt{x+2} - \sqrt{x-2} = \frac{1}{2}$. It follows that $x + 2 = (\frac{1}{2} + \sqrt{x-2})^2$, or $x + 2 = \frac{1}{4} + 2\sqrt{x-2} + x - 2$, or $\frac{15}{4} = \sqrt{x-2}$. It follows that $x - 2 = \frac{225}{16}$, from which we can find that $x = \boxed{\dfrac{257}{16}}$ as desired.

10. Let Q(x) $= x^2 + 2x + 3$, and suppose that $P(x)$ is a polynomial such that

$$P(Q(x)) = x^6 + 6x^5 + 18x^4 + 32x^3 + 35x^2 + 22x + 8.$$

Compute $P(2)$.

Solution: Notice that $P(2) = P(Q(Q^{-1}(2)))$. Therefore, it suffices to find the value of x such that $Q(x) = x^2 + 2x + 3 = 2$. This simplifies rather nicely, as we obtain $x^2 + 2x + 1 = (x + 1)^2 = 0 \rightarrow x = -1$. Therefore, our answer is $P(Q(-1)) = 1 - 6 + 18 - 32 + 35 - 22 + 8 = \boxed{2}$.

11. Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

Solution: Notice that we can rearrange the given inequality to obtain $(a - \frac{b}{2})^2 + (d - \frac{c}{2})^2 + \frac{3b^2}{4} + \frac{3c^2}{4} \geq \lambda bc$. To minimize the lefthand side of this inequality, we must have $a = \frac{b}{2}$ and $d = \frac{c}{2}$, giving us $\frac{3b^2}{4} + \frac{3c^2}{4} \geq \lambda bc$. By AM-GM, we know that $\frac{b^2+c^2}{2} \geq \sqrt{b^2c^2} = 1bc$. Therefore, $\lambda = \frac{\frac{3}{4}}{\frac{1}{2}} = \boxed{\dfrac{3}{2}}$ as desired.

12. Let a, b, c, x be reals with $(a + b)(b + c)(c + a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \text{ and } \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Compute x.

Solution: We can rewrite the given equations as $\frac{a^2}{a+b} - \frac{a^2}{a+c} = 20$, $\frac{b^2}{b+c} - \frac{b^2}{b+a} = 14$, and $\frac{c^2}{c+a} - \frac{c^2}{c+b} = x$. Adding these three equations and rearranging their terms, we get $\frac{b^2-c^2}{b+c} + \frac{c^2-a^2}{a+c} + \frac{a^2-b^2}{a+b} = 34 + x$. We can rewrite the Left Hand Side of this equation as $b - c + c - a + a - b = 0 = 34 + x$. Therefore, $x = \boxed{-34}$.

13. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

Solution: Adding the two equations, we get $a + b + \frac{1}{a} + \frac{1}{b} = (a + b)(1 + \frac{1}{ab}) = \frac{76}{15}$. Rearranging the two equations, we get $4b = ab + 1$ and $ab + 1 = \frac{16a}{15}$, which we can rewrite as $16b = 4ab + 4$ and $16a = 15ab + 15$. It follows that $a + b = \frac{19}{16}(ab + 1)$. It follows that $\frac{76}{15} = \frac{19}{16ab}(ab + 1)^2$, or $\frac{64ab}{15} = (ab + 1)^2$. Letting $ab = x$, we get that $(x + 1)^2 = \frac{64x}{15}$, or $x^2 - \frac{34x}{15} + 1 = 0$. By Vieta's equations, it follows that the product of all possible values of $x = ab$ is $\boxed{1}$.

14. Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11, P(11) = 17$, and $P(17) = 23$.

Solution: We can notice that when $x = 5$, $x = 11$, or $x = 17$, $P(x) = x + 6$. It follows that $P(x) = (x-5)(x-11)(x-17)Q(x)+x+6$. Because P is monic and its x^3 term has a coefficient of -29, it follows that $Q(x) = x+4$. Therefore, $P(x) = (x-5)(x-11)(x-17)(x+4)+x+6$. Because the sum of the coefficients of a polynomial P is $P(1)$, it follows that our answer is $(-4)\cdot(-10)\cdot(-16)\cdot 5+1+6 = 7 - 3200 = \boxed{-3193}$ as desired.

15. Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

Solution: We can notice that $f(x) = (x+3)^2-2$. It follows that $f(f(f(f(x)))) = (((x+3)^2+1)^2)+1)^2-2)$. Clearly, this will be minimized when $x = -3$, resulting in the value $(((1)^2+1)^2+1)^2-2 = 25 - 2 = \boxed{23}$ as desired.

3 Sources

1. 2008 November Harvard MIT Math Tournament General Problem 7
2. 2008 November Harvard MIT Math Tournament General Problem 9
3. 2008 November Harvard MIT Math Tournament General Problem 10
4. 2009 November Harvard MIT Math Tournament General Problem 2
5. 2009 November Harvard MIT Math Tournament General Problem 6
6. 2010 November Harvard MIT Math Tournament General Problem 5
7. 2010 November Harvard MIT Math Tournament General Problem 6
8. 2011 November Harvard MIT Math Tournament General Problem 1
9. 2011 November Harvard MIT Math Tournament General Problem 5
10. 2012 November Harvard MIT Math Tournament General Problem 2
11. 2013 November Harvard MIT Math Tournament General Problem 7
12. 2014 November Harvard MIT Math Tournament General Problem 8
13. 2016 November Harvard MIT Math Tournament General Problem 1
14. 2011 November Harvard MIT Math Tournament Team Problem 3
15. 2014 November Harvard MIT Math Tournament Team Problem 2