Algebra Handout 3

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1 Problems

1. Find all ordered pairs (x, y) such that $(x-2y)^2+(y-1)^2 = 0$.

2. Find the product of all real x for which $2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3} = 0$.

3. Find the largest positive integer n such that $n^3+4n^2-15n-18$ is the cube of an integer.

4. Given that $a+b+c = 5$ and that $1 \leq a, b, c \leq 2$, what is the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$?

5. Find the maximum value of $x+y$, given that $x^2+y^2-3y-1 = 0$.

6. A polynomial P is of the form $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$. Given that $P(2) = 27$, what is $P(3)$?

7. What is the sum of the positive solutions to $2x^2 - x \cdot \lfloor x \rfloor = 5$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x?

8. Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x+xy = 4$.

9. Find all real values of x for which

$$\frac{1}{\sqrt{x}+\sqrt{x-2}} + \frac{1}{\sqrt{x+2}+\sqrt{x}} = \frac{1}{4}.$$

10. Let Q(x) $= x^2+2x+3$, and suppose that $P(x)$ is a polynomial such that

$$P(Q(x)) = x^6+6x^5+18x^4+32x^3+35x^2+22x+8.$$

Compute $P(2)$.

11. Find the largest real number λ such that $a^2+b^2+c^2+d^2 \geq ab+\lambda bc+cd$ for all real numbers a, b, c, d.

12. Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \text{ and } \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Compute x.

13. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

14. Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11, P(11) = 17$, and $P(17) = 23$.

15. Let $f(x) = x^2+6x+7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

2 Sources

1. 2008 November Harvard MIT Math Tournament General Problem 7
2. 2008 November Harvard MIT Math Tournament General Problem 9
3. 2008 November Harvard MIT Math Tournament General Problem 10
4. 2009 November Harvard MIT Math Tournament General Problem 2
5. 2009 November Harvard MIT Math Tournament General Problem 6
6. 2010 November Harvard MIT Math Tournament General Problem 5
7. 2010 November Harvard MIT Math Tournament General Problem 6
8. 2011 November Harvard MIT Math Tournament General Problem 1
9. 2011 November Harvard MIT Math Tournament General Problem 5
10. 2012 November Harvard MIT Math Tournament General Problem 2
11. 2013 November Harvard MIT Math Tournament General Problem 7
12. 2014 November Harvard MIT Math Tournament General Problem 8
13. 2016 November Harvard MIT Math Tournament General Problem 1
14. 2011 November Harvard MIT Math Tournament Team Problem 3
15. 2014 November Harvard MIT Math Tournament Team Problem 2