Algebra Handout #6 Answers and Solutions

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1 Answers

1. $\frac{\sqrt{17}-1}{2}, \frac{\sqrt{13}+1}{2}$

2. 1490

3. $\frac{77}{8}$

4. 836

5. $\frac{3}{2}$

6. $1, \log_3 6$

7. 10

8. 4

9. $-\frac{1}{2}$

10. -336

11. $\frac{1}{1009}$

12. 4

13. -2

14. $-\frac{1}{9}$

15. 15

2 Solutions

1. Find x such that $\sqrt{c+\sqrt{c-x}} = x$ when $c = 4$.

Solution: Squaring both sides of this equation, we get $x^2 - c = \sqrt{c-x}$. Squaring the equation again, we get $x^4 - 2cx^2 + c^2 = c - x$. Plugging in $c = 4$, we get $x^4 - 8x^2 + x + 12 = 0$. Because the Rational Root Theorem does not produce any rational roots for this polynomial and its coefficients are not symmetric, we will attempt to factor this quartic as the product of two quadratics. This gives us

$$x^4 - 8x^2 + x + 12 = (x^2 + ax + b)(x^2 + cx + d)$$

Equating their coefficients, we get $a + c = 0$, $ac + b + d = -8$, $bc + ad = 1$, and $bd = 12$. It follows that $c = -a$, and therefore $b + d = a^2 - 8$ and $ad - ab = 1$. Notice that if $a = 1$, $d = -3$, and

$b = -4$, then everything works. It follows that we wish to solve

$$(x^2 + x - 4)(x^2 - x - 3) = 0$$

Using the quadratic formula on each of these quadratics gives us that either $x = \frac{1+\sqrt{13}}{2}$ or $x = \frac{\sqrt{17}-1}{2}$ as these are our only positive solutions. Because both are less than 4, it follows that our answer is $\boxed{\dfrac{\sqrt{17}-1}{2}, \dfrac{\sqrt{13}+1}{2}}$.

2. The tetranacci numbers are defined by the recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$ and $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. Given that $T_9 = 29$ and $T_{14} = 773$, calculate T_{15}.

Solution: Notice that $T_n + T_{n+1} + T_{n+2} = T_{n+3} - T_{n-1}$ and therefore $T_{n+4} = T_{n+3} - T_{n-1} + T_{n+3} = 2T_{n+3} - T_{n-1}$. Therefore, $T_{15} = 2T_{14} - T_{10}$. From here, we can find by brute force that $T_4 = 1, T_5 = 2, T_6 = 4, T_7 = 8, T_8 = 15$, and $T_{10} = 15 + 8 + 4 + 29 = 56$. It follows that our answer is $2 \cdot 773 - 56 = \boxed{1490}$.

3. What is the sum of the infinite series $\frac{20}{3} + \frac{17}{9} + \frac{20}{27} + \frac{17}{81} + \frac{20}{243} + \frac{17}{729} + ...$?

Solution: We can separate this infinite series into two infinite geometric series. First, we will calculate $\frac{20}{3} + \frac{20}{27} + \frac{20}{243} + \cdots$. Using the formula for the sum of an infinite geometric series, we get $\dfrac{\frac{20}{3}}{1 - \frac{1}{9}} = \frac{180}{24} = \frac{15}{2}$. Second, we will calculate $\frac{17}{9} + \frac{17}{81} + \frac{17}{729} + \cdots$. This is equal to $\dfrac{\frac{17}{9}}{1 - \frac{1}{9}} = \frac{17}{8}$. It follows that our answer is $\frac{15}{2} + \frac{17}{8} = \boxed{\dfrac{77}{8}}$.

4. If $xy = 15$ and $x + y = 11$, calculate the value of $x^3 + y^3$.

Solution: Notice that $x^3 + y^3 = (x+y)^3 - 3(xy)(x+y)$. It follows that our answer is $11^3 - 3 \cdot 11 \cdot 15 = \boxed{836}$.

5. Square S is the unit square with vertices at $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. We choose a random point (x, y) inside S and construct a rectangle with length x and width y. What is the average of $\lfloor p \rfloor$ where p is the perimeter of the rectangle? $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution: Notice that $p = 2x + 2y$ for the point (x, y). Therefore, we must do casework on when the value of $2x + 2y$ is between two integers. The probability that $1 \le 2x + 2y \le 2$ is $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$. The probability that $2 \le 2x + 2y \le 3$ is $\frac{7}{8} - \frac{1}{2} = \frac{3}{8}$. The probability that $3 \le 2x + 2y \le 4$ is $1 - \frac{7}{8} = \frac{1}{8}$. Therefore our answer is $\frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \boxed{\dfrac{3}{2}}$.

6. Find all solutions to $3^x - 9^{x-1} = 2$.

Solution: Let $3^x = y$. Then we have that $y - \frac{y^2}{9} = 2$, or $y^2 - 9y + 18 = 0$. It follows that either $y = 3$, or $y = 6$, and therefore our solutions are $x = \boxed{1, \log_3 6}$.

7. Find the value of

$$\frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + \cdots.$$

Solution: We wish to calculate

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{2\binom{n}{2} + n}{2^n} = 2\sum_{n=1}^{\infty} \frac{\binom{n}{2}}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n}$$

Now notice that if we expand $(1+x+x^2+\cdots)^2$, we get $1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots$, and if we expand $(1+x+x^2+\cdots)^3$, we get $1+3x+6x^2+10x^3+\cdots+\binom{n}{2}x^{n-2}+\cdots$. Notice that $(1+x+x^2+\cdots)=\frac{1}{1-x}$. It follows that

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{1}{2}\cdot(\frac{1}{1-\frac{1}{2}})^2=2$$

and that

$$2\sum_{n=1}^{\infty}\frac{\binom{n}{2}}{2^n}=2\cdot\frac{1}{2}\cdot(\frac{1}{1-\frac{1}{2}})^3=8$$

It follows that our answer is $2+8=\boxed{10}$.

8. Find the value of y such that the following equation has exactly three solutions.

$$||x-1|-4|=y.$$

Solution: Notice that when $y=4$, the solutions for x will be the solutions to $|x-1|=0$ and $|x-1|=8$. Clearly, there will be 3 solutions to these equations, so it follows that our answer is $y=\boxed{4}$.

9. Compute

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}.$$

Solution: Notice that $\frac{1}{(2k+1)(2k+3)}-\frac{1}{(2k-1)(2k+1)}=\frac{-4}{(2k-1)(2k+1)(2k+3)}$. It follows that we wish to calculate

$$-4\sum_{k=1}^{\infty}\frac{1}{(4k-3)(4k-1)(4k+1)}=-1\sum_{k=1}^{\infty}\left\{\frac{1}{(4k-1)(4k-3)}-\frac{1}{(4k-1)(4k+1)}\right\}$$

This simplification results from partial fraction decomposition. Using this decomposition again, we get that we wish to calculate

$$-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{4k-3}-\frac{1}{4k+1}=-\frac{1}{2}\cdot(1-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\frac{1}{9}-\cdots)=\boxed{-\frac{1}{2}}$$

10. Consider the function $f(x,y,z)=(x-y+z,y-z+x,z-x+y)$ and denote by $f^{(n)}(x,y,z)$ the function f applied n times to the tuple (x,y,z). Let r_1,r_2,r_3 be the three roots of the equation $x^3-4x^2+12=0$ and let $g(x)=x^3+a_2x^2+a_1x+a_0$ be the cubic polynomial with the tuple $f^{(3)}(r_1,r_2,r_3)$ as roots. Find the value of a_1.

Solution: By Brute Force, we can find that $f^{(3)}(r_1,r_2,r_3)=(3r_2+3r_3-5r_1,3r_1+3r_3-5r_2,3r_1+3r_2-5r_3)$. By Vieta's Formulas, we know that

$$a_1=(3r_2+3r_3-5r_1)(3r_1+3r_3-5r_2)+(3r_2+3r_3-5r_1)(3r_1+3r_2-5r_3)+(3r_1+3r_2-5r_3)(3r_1+3r_3-5r_2)$$

Because $r_1+r_2+r_3=4$, we can rewrite this as

$$a_1=(12-8r_1)(12-8r_2)+(12-8r_1)(12-8r_3)+(12-8r_2)(12-8r_3)$$

Expanding this, we get that we wish to calculate

$$a_1 = 432 - 192(r_1 + r_2 + r_3) + 64(r_1r_2 + r_1r_3 + r_2r_3)$$

From here, we can easily find that $a_1 = 432 - 192 \cdot 4 + 64 \cdot 0 = \boxed{-336}$.

11. A function f with its domain on the positive integers $\mathbb{N} = \{1, 2, ...\}$ satisfies the following conditions:
(a): $f(1) = 2017$.
(b): $\sum_{i=1}^{n} f(i) = n^2 f(n)$, for every positive integer $n > 1$.
What is the value of $f(2017)$?

Solution: Notice that by the given condition, $f(n) = n^2 f(n) - (n-1)^2 f(n-1)$ for all $n \geq 2$. It follows that $f(n) = \frac{n^2-2n+1}{n^2-1} \cdot f(n-1)$ for all $n \geq 2$. It follows that we wish to calculate

$$2017 \cdot \prod_{n=2}^{2017} \frac{n^2-2n+1}{n^2-1} = 2017 \cdot \prod_{n=2}^{2017} \frac{n-1}{n+1} = 2017 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \cdots \cdot \frac{2015}{2017} \cdot \frac{2016}{2018}$$

This clearly telescopes to $2017 \cdot \frac{1}{2017} \cdot \frac{2}{2018} = \boxed{\dfrac{1}{1009}}$.

12. Suppose that there is a set of 2016 positive numbers, such that both their sum, and the sum of their reciprocals are equal to 2017. Let x be one of those numbers. Find the maximum possible value of $x + \frac{1}{x}$.

Solution: Let the numbers be x_i as i ranges from 1 to 2016. Notice that if we add the numbers to their reciprocals, then we get

$$\sum_i x_i + \frac{1}{x_i} = 4034$$

By the Arithmetic Mean - Geometric Mean Inequality, we know that $\dfrac{x_i + \frac{1}{x_i}}{2} \geq \sqrt{x_i \cdot \frac{1}{x_i}} = 1$. Therefore, $x_i + \frac{1}{x_i} \geq 2$. To maximize the value of $x_i + \frac{1}{x_i}$ for some i, we should make $x_y + \frac{1}{x_y}$ 2 for all $y \neq i$. In this case, it follows that $x_i + \frac{1}{x_i} = 4034 - 2015 \cdot 2 = \boxed{4}$.

13. Define the operation $a@b$ to be $3 + ab + a + 2b$. There exists a number x such that $x@b = 1$ for all b. Find x.

Solution: Plugging in (x, b) for (a, b), we get $3 + xb + x + 2b = 1$ for all b. It follows that $(x+2)(b+1) = 0$. Note that if $x = -2$, this equation will be true for every value of b. Therefore, $x = \boxed{-2}$.

14. The distinct rational numbers $-\sqrt{-x}, x$, and $-x$ form an arithmetic sequence in that order. Determine the value of x.

Solution: By the given assertion, we know that $2x = -\sqrt{-x} - x$, or $3x = -\sqrt{-x}$. Squaring, we get $9x^2 = -x$. It follows that either $x = 0$ or $x = -\frac{1}{9}$. Because x and $-x$ are distinct, it follows that our answer is $x = \boxed{-\dfrac{1}{9}}$.

15. Let A, B, and k be integers, where k is positive and the greatest common divisor of A, B, and k is 1. Define $x\#y$ by the formula $x\#y = \dfrac{Ax + By}{kxy}$. If $8\#4 = \frac{1}{2}$ and $3\#1 = \frac{13}{6}$, determine the sum $A + B + k$.

Solution: By the first assertion, we know that $16k = 8A + 4B$ and by the second assertion, we know that $\frac{3k}{2} = 3A + B$ or $3k = 6A + 2B$. Subtracting the second equation from the first, we get that $2A + 2B = 13k$. Subtracting this equation twice from the first, we get $4A = -10k$ or $2A = -5k$. Because $\gcd(A, k) = 1$ and k is positive, it follows that $k = 2$ and $A = -5$. It follows that $B = 18$. Therefore $A + B + k = -5 + 18 + 2 = \boxed{15}$.

3 Sources

1. 2016 Berkeley Math Tournament Fall Team Problem 20
2. 2016 Berkeley Math Tournament Fall Team Problem 15
3. 2017 Berkeley Math Tournament Spring Individual Problem 7
4. 2017 Berkeley Math Tournament Spring Individual Problem 8
5. 2017 Berkeley Math Tournament Spring Individual Problem 12
6. 2017 Berkeley Math Tournament Spring Analysis Problem 2
7. 2017 Berkeley Math Tournament Spring Analysis Problem 4
8. 2017 Berkeley Math Tournament Spring Analysis Problem 5
9. 2017 Berkeley Math Tournament Spring Analysis Problem 7
10. 2017 Berkeley Math Tournament Spring Analysis Problem 6
11. 2017 Berkeley Math Tournament Spring Team Problem 8
12. 2017 Berkeley Math Tournament Spring Team Problem 14
13. 2017 Berkeley Math Tournament Fall Individual Problem 9
14. 2017 Berkeley Math Tournament Fall Individual Problem 11
15. 2017 Berkeley Math Tournament Fall Individual Problem 17