Algebra Handout #6

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1 Problems

1. Find x such that $\sqrt{c+\sqrt{c-x}} = x$ when $c = 4$.

2. The tetranacci numbers are defined by the recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$ and $T_0 = T_1 = T_2 = 0$ and $T_3 = 1$. Given that $T_9 = 29$ and $T_{14} = 773$, calculate T_{15}.

3. What is the sum of the infinite series $\frac{20}{3} + \frac{17}{9} + \frac{20}{27} + \frac{17}{81} + \frac{20}{243} + \frac{17}{729} + ...$?

4. If $xy = 15$ and $x + y = 11$, calculate the value of $x^3 + y^3$.

5. Square S is the unit square with vertices at $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. We choose a random point (x,y) inside S and construct a rectangle with length x and width y. What is the average of $\lfloor p \rfloor$ where p is the perimeter of the rectangle? $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

6. Find all solutions to $3^x - 9^{x-1} = 2$.

7. Find the value of

$$\frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + \cdots.$$

8. Find the value of y such that the following equation has exactly three solutions.

$$||x-1|-4| = y.$$

9. Compute

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)}.$$

10. Consider the function $f(x,y,z) = (x-y+z, y-z+x, z-x+y)$ and denote by $f^{(n)}(x,y,z)$ the function f applied n times to the tuple (x,y,z). Let r_1, r_2, r_3 be the three roots of the equation $x^3 - 4x^2 + 12 = 0$ and let $g(x) = x^3 + a_2x^2 + a_1x + a_0$ be the cubic polynomial with the tuple $f^{(3)}(r_1, r_2, r_3)$ as roots. Find the value of a_1.

11. A function f with its domain on the positive integers $\mathbb{N} = \{1, 2, ...\}$ satisfies the following conditions:
(a): $f(1) = 2017$.
(b): $\sum_{i=1}^{n} f(i) = n^2 f(n)$, for every positive integer $n > 1$.
What is the value of $f(2017)$?

12. Suppose that there is a set of 2016 positive numbers, such that both their sum, and the sum of their reciprocals are equal to 2017. Let x be one of those numbers. Find the maximum possible value of $x + \frac{1}{x}$.

13. Define the operation $a@b$ to be $3 + ab + a + 2b$. There exists a number x such that $x@b = 1$ for all b. Find x.

14. The distinct rational numbers $-\sqrt{-x}, x$, and $-x$ form an arithmetic sequence in that order. Determine the value of x.

15. Let A, B, and k be integers, where k is positive and the greatest common divisor of A, B, and k is 1. Define $x\#y$ by the formula $x\#y = \dfrac{Ax + By}{kxy}$. If $8\#4 = \frac{1}{2}$ and $3\#1 = \frac{13}{6}$, determine the sum $A + B + k$.

2 Sources

1. 2016 Berkeley Math Tournament Fall Team Problem 20
2. 2016 Berkeley Math Tournament Fall Team Problem 15
3. 2017 Berkeley Math Tournament Spring Individual Problem 7
4. 2017 Berkeley Math Tournament Spring Individual Problem 8
5. 2017 Berkeley Math Tournament Spring Individual Problem 12
6. 2017 Berkeley Math Tournament Spring Analysis Problem 2
7. 2017 Berkeley Math Tournament Spring Analysis Problem 4
8. 2017 Berkeley Math Tournament Spring Analysis Problem 5
9. 2017 Berkeley Math Tournament Spring Analysis Problem 7
10. 2017 Berkeley Math Tournament Spring Analysis Problem 6
11. 2017 Berkeley Math Tournament Spring Team Problem 8
12. 2017 Berkeley Math Tournament Spring Team Problem 14
13. 2017 Berkeley Math Tournament Fall Individual Problem 9
14. 2017 Berkeley Math Tournament Fall Individual Problem 11
15. 2017 Berkeley Math Tournament Fall Individual Problem 17