Algebra Handout # 7 Answers and Solutions

Walker Kroubalkian

April 3, 2018

1 Answers

1. \$2.60

2. 12

3. $\frac{13}{7}$

4. 2019

5. 18

6. 113

7. 36

8. 3

9. 4

10. $\frac{11}{14}$

11. 40

12. 18

13. $\sqrt[6]{3}$

14. $\frac{1023}{1024}$

15. $-\frac{9}{4}$

2 Solutions

1. At the grocery store, 3 avocados and 2 pineapples cost \$8.80, while 5 avocados and 3 pineapples cost \$14.00. How much do 1 avocado and 1 pineapple cost in dollars?

Solution: Let the cost of an avocado be a and let the cost of a pineapple be p. It follows that $3a + 2p = 8.80$ and $5a + 4p = 14.00$. Subtracting the first equation from the second tells us that $2a + 2p = 5.20$. Dividing this equation by 2 tells us that the cost of 1 avocado and 1 pineapple is $\boxed{\$2.60}$.

2. Let a, b, c, d be an increasing sequence of numbers such that a, b, c forms a geometric sequence and b, c, d forms an arithmetic sequence. Given that $a = 8$ and $d = 24$, what is b?

Solution: By the given information, we know that $b^2 = ac$ and $b + d = 2c$. Plugging the second equation into the first tells us that $b^2 = \frac{a(b+d)}{2}$. It follows that $2b^2 = 8b + 192$, or $b^2 = 4b + 96$. This equation can be factored as $(b-12)(b+8) = 0$. Because this is an increasing sequence, it follows that $b = \boxed{12}$.

3. Given that the roots of the polynomial $x^3 - 7x^2 + 13x - 7 = 0$ are r, s, t, compute the value of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.

Solution: Notice that we wish to calculate $\frac{rs+rt+st}{rst}$. By Vieta's Formulas, we know that $rst = 7$ and $rs + rt + st = 13$. Therefore our answer is $\boxed{\frac{13}{7}}$.

4. Let $a_1, a_2, a_3, a_4, a_5, ...$ be a geometric progression with positive ratio such that $a_1 > 1$ and $(a_{1357})^3 = a_{34}$. Find the smallest integer n such that $a_n < 1$.

Solution: Let $a_1 = A$ and let the common ratio be r. It follows that $a_{1357} = Ar^{1356}$ and $a_{34} = Ar^{33}$. It follows that $A^3r^{4068} = Ar^{33}$, or $A^2 = r^{-4035}$. It follows that $A = r^{-\frac{4035}{2}}$. Clearly, we must have that $r < 1$, so it follows that $Ar^{2018} = r^{\frac{4036}{2} - \frac{4035}{2}} = r^{\frac{1}{2}}$ is the first term which is less than 1, and therefore our answer is $n = \boxed{2019}$.

5. Let x, y, z be non-negative real numbers satisfying $xyz = \frac{2}{3}$. Compute the minimum value of

$$x^2 + 6xy + 18y^2 + 12yz + 4z^2.$$

Solution: This expression can be rewritten as $(x+3y)^2 + (3y+2z)^2$. By the Cauchy Schwarz Inequality, we must have that $((x+3y)^2 + (3y+2z)^2) \cdot (1^2 + 1^2) \geq (x + 3y + 3y + 2z)^2$. Therefore, we wish to find the minimum value of $\frac{(x+6y+2z)^2}{2}$. By the AM-GM inequality, we know that $\frac{x+6y+2z}{3} \geq \sqrt[3]{12xyz} = 2$. It follows that the minimum value of our expression is $\frac{(2 \cdot 3)^2}{2} = \boxed{18}$.

6. Define $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. If $| \ x \ | \leq 8$, find the number of real solutions to the equation

$$\{x\} + \left\{x^2\right\} = 1.$$

Solution: Let $x = i + f$ where i is an integer and $0 \leq f < 1$. It follows that the given equation is equivalent to $\left\{i^2 + 2if + f^2\right\} = \left\{2if + f^2\right\} = 1 - f$. This statement is equivalent to letting $2if + f^2 - (1-f)$ be an integer, or letting $2if + f^2 + f$ be an integer. From here, it is a simple matter of plugging in every integer i in the range $-8 \leq i < 8$ and solving for fractional parts f. When $i = -8$, we want $f^2 - 15f$ to be an integer. By the intermediate value theorem (IVT), this will take on every value between 0 and -14 with the exceptions of 0 and -14 as they correspond to when $f = 0$ and $f = 1$, respectively, so we have 13 solutions when $i = -8$. When $i = -7$, we want $f^2 - 13f$ to be an integer. We can easily find that there are 11 solutions in this case. Continuing this pattern, we find that there are a total of $13+11+9+7+5+3+1+0+1+3+5+7+9+11+13+15 = 7^2 + 8^2 = \boxed{113}$.

7. Katy only owns two types of books: comic books and nature books. $\frac{1}{3}$ of her books are comic books. After going to a booksale, she buys 20 more comic books, so $\frac{4}{7}$ of her books are now comic books. How many books did she have originally?

Solution: Let Katy own c comic books and b total books before the sale. It follows that $\frac{c}{b} = \frac{1}{3}$ and $\frac{c+20}{b+20} = \frac{4}{7}$. It follows that $b = 3c$ and $7c + 140 = 4b + 80$. Substituting the first equation in the second tells us that $5c = 60$, or $c = 12$. Therefore, $b = 3 \cdot 12 = \boxed{36}$.

8. Let $f(x) = x^3 - n_1x^2 + (n_2 - k^2)x - (n_3 - k^4)$. Suppose that n_1, n_2, and n_3 form a geometric sequence with common ratio k and that the roots of f are nonzero and form an arithmetic sequence with common difference also k. Find k.

Solution: Let $n_1 = N$ and let the roots be a, $a-k$, and $a+k$. It follows that $3a = N$, $3a^2 - k^2 = Nk - k^2$, and $a^3 - ak^2 = Nk^2 - k^4$. Plugging the first equation into the other two equation gives us that $3a^2 - k^2 = 3ak - k^2$ and $a^3 - ak^2 = 3ak^2 - k^4$, or $3a^2 = 3ak$ and $a^3 = 4ak^2 - k^4$. The first equation tells us that $a = k$, and plugging this into the second equation tells us that $k^3 = 4k^3 - k^4$. Because k is nonzero, we know that $k = \boxed{3}$.

9. If a, b, c are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.

Solution: Notice that the given expression is equivalent to $(c-b)(a-c)$. If we let $c - b = x$ and $a - c = y$, then we have that $x + y = c - b + a - c = a - b = 4$. Because it is impossible for both x and y to be negative, their product will be maximized when each of x and y are non-negative. It follows by AM-GM that the maximum value of xy is $(\frac{x+y}{2})^2 = 2^2 = \boxed{4}$.

10. If $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ and $\frac{1}{x+1} + \frac{1}{y+1} = \frac{3}{8}$, compute $\frac{1}{x-1} + \frac{1}{y-1}$.

Solution: We know $\frac{x+y}{xy} = \frac{1}{2}$ and $\frac{x+y+2}{xy+x+y+1} = \frac{3}{8}$, and we wish to calculate $\frac{x+y-2}{xy-x-y+1}$. From the given equations, we know that $2x + 2y = xy$ and $8x + 8y + 16 = 3xy + 3x + 3y + 3$. Substituting the second equation in the first gives us $x + y = 13$, and it follows that $xy = 26$. Therefore our answer is $\frac{13-2}{26-13+1} = \boxed{\frac{11}{14}}$.

11. Connie owns a small farm and grows mangos and pineapples. After one harvest she increased her mango supply by 50% but also sold half of her pineapples. Given that she has a net loss of 10 fruit after the harvest, and that she has the same number of mangos as pineapples after the harvest, how much fruit did she initially have?

Solution: Let Connie's intial number of mangos and initial number of pineapples be m and p, respectively. We must have that $\frac{3}{2}m + \frac{1}{2}p - (m + p) = \frac{m-p}{2} = -10$ and that $\frac{3}{2}m = \frac{1}{2}p$. The first equation tells us that $m = p - 20$ and the second equation tells us that $3m = p$. It follows that $2m = 20$ and $m = 10$, and therefore $p = 30$. It follows that she initially had $30 + 10 = \boxed{40}$ fruit.

12. For some real number c, the graphs of the equation $y =\mid x - 20 \mid + \mid x + 18 \mid$ and the line $y = x + c$ intersect at exactly one point. What is c?

Solution: Notice that the given function has a slope of -2 when $x < -18$, the function has a slope of 0 when $-18 < x < 20$, and the function has a slope of 2 when $20 < x$. It follows that because the function's rate of change is non decreasing, we must have that our line $y = x + c$ intersects it at the point where its slope becomes greater than 1. It follows that the two graphs must intersect at $x = 20$, and this gives us the intersection point $(20, 38)$. It follows that $c = 38 - 20 = \boxed{18}$.

13. Compute the positive real number x satisfying

$$x^{2x^6} = 3.$$

Solution: We will begin by taking $\log_3$ of both sides. Doing so gives us the equation $2x^6 \cdot \log_3 x = 1$. If we let $\log_3 x = a$, then we know that $x = 3^a$ and it follows that $2a \cdot 3^{6a} = 1$. By inspection, we can notice that if $2a = \frac{1}{3}$, then the equation is true. It follows that $a = \frac{1}{6}$, and therefore our answer is $3^{\frac{1}{6}} = \boxed{\sqrt[6]{3}}$.

14. John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

Solution: Notice that each time this operation is performed, exactly $\frac{1}{2}$ of the orange juice is lost. Therefore, at the end of the 10 operations, only $(\frac{1}{2})^{10} = \frac{1}{1024}$ of the orange juice remains, and it follows that our answer is $\boxed{\dfrac{1023}{1024}}$.

15. Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

Solution: Let $x = 2^{2^a}$ for some real number a. Then we have that $a - 1 + \frac{a-2}{2} + \frac{a}{4} = 0$ or $7a = 8$ or $a = \frac{7}{8}$. We wish to calculate $a - 4 + \frac{a-1}{4} + \frac{a}{2}$. Plugging in $a = \frac{7}{8}$, we get an answer of $\frac{7a-17}{4} = \boxed{-\dfrac{9}{4}}$.

3 Sources

1. 2018 Stanford Math Tournament Algebra Problem 1
2. 2018 Stanford Math Tournament Algebra Problem 2
3. 2018 Stanford Math Tournament Algebra Problem 3
4. 2018 Stanford Math Tournament Algebra Problem 5
5. 2018 Stanford Math Tournament Algebra Problem 7
6. 2018 Stanford Math Tournament Algebra Problem 8
7. 2018 Stanford Math Tournament General Problem 4
8. 2018 Stanford Math Tournament General Problem 25
9. 2018 Stanford Math Tournament Algebra Tiebreaker Problem 1
10. 2018 Stanford Math Tournament Algebra Tiebreaker Problem 2
11. 2018 Stanford Math Tournament General Tiebreaker Problem 1
12. 2018 Harvard MIT Math Tournament February Algebra and Number Theory Problem 1
13. 2018 Harvard MIT Math Tournament February Algebra and Number Theory Problem 2
14. 2018 Harvard MIT Math Tournament February Guts Problem 2
15. 2018 Harvard MIT Math Tournament February Guts Problem 8