

## Algebra Handout # 7

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## 1 Problems

- At the grocery store, 3 avocados and 2 pineapples cost \$8.80, while 5 avocados and 3 pineapples cost \$14.00. How much do 1 avocado and 1 pineapple cost in dollars?
- Let  $a, b, c, d$  be an increasing sequence of numbers such that  $a, b, c$  forms a geometric sequence and  $b, c, d$  forms an arithmetic sequence. Given that  $a = 8$  and  $d = 24$ , what is  $b$ ?
- Given that the roots of the polynomial  $x^3 - 7x^2 + 13x - 7 = 0$  are  $r, s, t$ , compute the value of  $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ .
- Let  $a_1, a_2, a_3, a_4, a_5, \dots$  be a geometric progression with positive ratio such that  $a_1 > 1$  and  $(a_{1357})^3 = a_{34}$ . Find the smallest integer  $n$  such that  $a_n < 1$ .
- Let  $x, y, z$  be non-negative real numbers satisfying  $xyz = \frac{2}{3}$ . Compute the minimum value of

$$x^2 + 6xy + 18y^2 + 12yz + 4z^2.$$

- Define  $\{x\} = x - \lfloor x \rfloor$ , where  $\lfloor x \rfloor$  denotes the largest integer not exceeding  $x$ . If  $|x| \leq 8$ , find the number of real solutions to the equation

$$\{x\} + \{x^2\} = 1.$$

- Katy only owns two types of books: comic books and nature books.  $\frac{1}{3}$  of her books are comic books. After going to a booksale, she buys 20 more comic books, so  $\frac{4}{7}$  of her books are now comic books. How many books did she have originally?
- Let  $f(x) = x^3 - n_1x^2 + (n_2 - k)x - (n_3 - k^4)$ . Suppose that  $n_1, n_2$ , and  $n_3$  form a geometric sequence with common ratio  $k$  and that the roots of  $f$  are nonzero and form an arithmetic sequence with common difference also  $k$ . Find  $k$ .
- If  $a, b, c$  are real numbers with  $a - b = 4$ , find the maximum value of  $ac + bc - c^2 - ab$ .
- If  $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$  and  $\frac{1}{x+1} + \frac{1}{y+1} = \frac{3}{8}$ , compute  $\frac{1}{x-1} + \frac{1}{y-1}$ .
- Connie owns a small farm and grows mangos and pineapples. After one harvest she increased her mango supply by 50% but also sold half of her pineapples. Given that she has a net loss of 10 fruit after the harvest, and that she has the same number of mangos as pineapples after the harvest, how much fruit did she initially have?
- For some real number  $c$ , the graphs of the equation  $y = |x - 20| + |x + 18|$  and the line  $y = x + c$  intersect at exactly one point. What is  $c$ ?
- Compute the positive real number  $x$  satisfying

$$x^{2x^6} = 3.$$

**14.** John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

**15.** Suppose a real number  $x > 1$  satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

## 2 Sources

1. 2018 Stanford Math Tournament Algebra Problem 1
2. 2018 Stanford Math Tournament Algebra Problem 2
3. 2018 Stanford Math Tournament Algebra Problem 3
4. 2018 Stanford Math Tournament Algebra Problem 5
5. 2018 Stanford Math Tournament Algebra Problem 7
6. 2018 Stanford Math Tournament Algebra Problem 8
7. 2018 Stanford Math Tournament General Problem 4
8. 2018 Stanford Math Tournament General Problem 25
9. 2018 Stanford Math Tournament Algebra Tiebreaker Problem 1
10. 2018 Stanford Math Tournament Algebra Tiebreaker Problem 2
11. 2018 Stanford Math Tournament General Tiebreaker Problem 1
12. 2018 Harvard MIT Math Tournament February Algebra and Number Theory Problem 1
13. 2018 Harvard MIT Math Tournament February Algebra and Number Theory Problem 2
14. 2018 Harvard MIT Math Tournament February Guts Problem 2
15. 2018 Harvard MIT Math Tournament February Guts Problem 8