Algebra Handout 1 Answers and Solutions

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1 Answers

1. 6

2. -$\frac{1}{3}$

3. 40

4. 999999

5. 1

6. 16

7. $\frac{10}{11}$

8. $\frac{25}{4}$

9. 40

10. -353

11. 15

12. $\frac{68}{3}$

13. 8960

14. 168

15. 10

2 Solutions

1. Compute the unique positive integer that, when squared, is equal to six more than five times itself.

Solution: Let the integer be x. We have $x^2 = 5x + 6$. Factoring, we get $(x - 6)(x + 1) = 0$. This gives solution of $x = -1$ and $x = 6$. Because x is positive, our answer is $\boxed{6}$.

2. An infinite geometric sequence has a first term of 12, and all terms in the sequence sum to 9. Compute the common ratio between consecutive terms of the geometric sequence.

Solution: Let the common ratio be r. Using the fact that the sum of the infinite geometric sequence $a + ar + ar^2 + \cdots$ is $\frac{a}{1-r}$. Letting this be equal to 9 and plugging in 12 for a gives us

$$\frac{12}{1-r} = 9 \rightarrow 12 = 9 - 9r \rightarrow \boxed{-\frac{1}{3}}$$

3. Alice and Bob are painting a house. If Alice and Bob do not take any breaks, they will finish painting the house in 20 hours. If however, Bob stops painting once the house is half-finished, then the house takes 30 hours to finish. Given that Alice and Bob paint at a constant rate, compute how many hours it will take for Bob to paint the entire house if he does it by himself.

Solution: Notice that if Alice and Bob together take 20 hours to paint the entire house, they together will paint half of the house in $\frac{20}{2} = 10$ hours. It follows that it takes Alice $30 - 10 = 20$ hours to paint half of the house, or it takes her $20 \times 2 = 40$ hours to paint the entire house. Let Bob take b hours to paint the entire house. It follows that in one hour, they together paint $\frac{1}{40} + \frac{1}{b}$ of the house. However, we know that together they take 20 hours to paint the house, so in one hour, they will paint $\frac{1}{20}$ of the house. Therefore,

$$\frac{1}{40} + \frac{1}{b} = \frac{1}{20} \to \frac{1}{b} = \frac{1}{40} \to b = \boxed{40}$$

4. Compute $9^6 + 6 \cdot 9^5 + 15 \cdot 9^4 + 20 \cdot 9^3 + 15 \cdot 9^2 + 6 \cdot 9$.

Solution: Notice that if $(9 + 1)^6$ is expanded using the binomial theorem, we get

$$(9+1)^6 = 9^6 + \binom{6}{1}9^5 + \binom{6}{2}9^4 + \cdots + 1 = 9^6 + 6 \cdot 9^5 + 15 \cdot 9^4 + \cdots + 6 \cdot 9 + 1$$

This is one more than the sum we wish to compute, so our answer is $(9 + 1)^6 - 1 = 1000000 - 1 = \boxed{999999}$ as desired.

5. Let x_1 and x_2 be the roots of $x^2 - x - 2014$, with $x_1 < x_2$. Let x_3 and x_4 be the roots of $x^2 - 2x - 2014$, with $x_3 < x_4$. Compute $(x_4 - x_2) + (x_3 - x_1)$.

Solution: Notice that the result we wish to compute is $(x_4+x_3)-(x_2+x_1)$. By Vieta's Formula's, if p and q are roots of $a^2+b\cdot x+c$, then $p+q = -\frac{b}{a}$ and $p\cdot q = \frac{c}{a}$. It follows that $x_1+x_2 = -(-1) = 1$ and $x_3 + x_4 = -(-2) = 2$, so our answer is $2 - 1 = \boxed{1}$ as desired.

6. Robin goes birdwatching one day. He sees three types of birds: penguins, pigeons, and robins. $\frac{2}{3}$ of the birds he sees are robins. $\frac{1}{8}$ of the birds he sees are penguins. He sees exactly 5 pigeons. How many robins does Robin see?

Solution: The fraction of the birds which are pigeons must be $1 - \frac{2}{3} - \frac{1}{8} = \frac{5}{24}$. Letting the total number of birds be T, we have $\frac{5}{24}T = 5 \to T = 24$. It follows that Robin saw $\frac{2}{3} \times 24 = \boxed{16}$ robins as desired.

7. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

Solution: Notice that each day, if the tree has y pounds of apples at dawn, it will go to $1.1\cdot(y-x)$ by the next dawn. In other words, the tree increases by $0.1\cdot y - 1.1\cdot x$ pounds each day. Clearly, this number increases as y increases and decreases as y decreases. For this reason, if for whatever the reason the number of pounds on the tree ever decreases, it will continue to decrease at a growing rate. It follows that if the tree ever decreases in weight, it will continue to decrease until it runs out of weight. As a result, the tree cannot decrease in weight from its initial weight of 10 pounds. Therefore to maximize the value of x, x should be chosen such that after every dawn, the weight of the tree is exactly 10 pounds. Plugging in $y = 10$ into the "formula" above gives us:

$$1.1 \cdot (y - x) = y \to 1.1 \cdot (10 - x) = 10 \cdot x = \boxed{\frac{10}{11}}$$

8. What is the greatest possible value of c such that $x^2 + 5x + c = 0$ has at least one real solution?

Solution 1: Let the roots of this quadratic be p and q. By Vieta's formulas, we have $-5 = p + q$ and $c = p \cdot q$. Notice that in order to maximize c, p and q should have the sign to make c positive. Therefore, p and q should both be negative. Let $p' = -p$ and $q' = -q$. It follows that $p' + q' = 5$ and we wish to maximize $p'q'$. By the Arithmetic Mean-Geometric Mean Inequality (AM-GM), we have that:

$$\frac{p' + q'}{2} \geq \sqrt{p'q'} \to \boxed{\frac{25}{4}} \geq p'q'$$

as desired.

Solution 2: Let the roots of this quadratic be p and q. By Vieta's formulas, we have $c = p \cdot q$. In order for this quadratic to have real roots, we must have that its discriminant is nonnegative. It follows that $5^2 - 4 \cdot c \geq 0$. Rearranging this, we get $\boxed{\frac{25}{4}} \geq c$ as desired.

9. Caroline wants to plant 10 trees in her orchard. Planting n apple trees requires n^2 square meters, planting n apricot trees requires $5n$ square meters, and planting n plum trees requires n^3 square meters. If she is committed to growing only apple, apricot, and plum trees, what is the least amount of space, in square meters, that her garden will take up?

Solution: Notice that it is not worth it to plant more than 1 plum tree, as adding a second plum tree requires 7 more square meters while adding an additional apricot tree always requires only 5 more square meters. In addition, notice it is not worth to plant more than 3 apple trees, as adding a fourth apple tree requires an additional $4^2 - 3^2 = 7$ square meters. Therefore, the minimum amount of space her garden will take up is $3^2 + 1^3 + 5 \cdot (10 - 3 - 1) = \boxed{40}$ as desired.

10. Let a and b be the solutions to $x^2 - 7x + 17 = 0$. Compute $a^4 + b^4$.

Solution: By Vieta's Formulas, we have $a + b = 7$ and $ab = 17$. Notice that

$$a^2 + b^2 = (a + b)^2 - 2ab = 7^2 - 2 \times 17 = 15$$

In addition, notice

$$(a^2 + b^2)^2 - 2a^2b^2 = 15^2 - 2(17)^2 = \boxed{-353}$$

11. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

Solution: Let the length of one lap around the circuit be x miles. Remembering that $r = \frac{d}{t}$ where r is rate, d is distance, and t is time, we get that each of the first three laps takes $\frac{x}{9}$ hours to complete. If the rate on the fourth lap is z mph, than we have that the fourth lap takes $\frac{x}{z}$ hours to complete. Therefore, the total time spent is $3 \times \frac{x}{9} + \frac{x}{z} = \frac{x}{3} + \frac{x}{z}$. It follows by $r = \frac{d}{t}$ that

$$\frac{4x}{\frac{x}{3} + \frac{x}{z}} = \frac{4}{\frac{1}{3} + \frac{1}{z}} = 10 \to \frac{1}{z} = \frac{2}{5} - \frac{1}{3} \to z = \boxed{15}$$

12. Given that $f(x) + 2f(8-x) = x^2$ for all real x, compute $f(2)$.

Solution 1: Plugging in $x = 2$ to the equation gives us

$$f(2) + 2f(6) = 4$$

Plugging in $x = 6$ to the equation gives us

$$f(6) + 2f(2) = 36$$

Letting $f(2) = a$ and $f(6) = b$, we wish to solve $a + 2b = 4$ and $b + 2a = 36$. Solving, we get $a = f(2) = 36 - \frac{40}{3} = \boxed{\frac{68}{3}}$ as desired.

Solution 2: The given equation is

$$f(x) + 2f(8-x) = x^2$$

Replacing x with $8 - x$ gives us

$$f(8-x) + 2f(x) = (8-x)^2$$

Adding these equations and dividing by 3 gives us

$$f(x) + f(8-x) = \frac{2x^2 - 16x + 64}{3}$$

Subtracting this from the second equation gives us $f(x) = \frac{x^2-32x+128}{3}$. Plugging in $x = 2$ gives us $f(2) = \boxed{\frac{68}{3}}$ as desired.

13. Karl likes the number 17. His favorite polynomials are monic quadratics with integer coefficients such that 17 is a root of the quadratic and the roots differ by no more than 17. Compute the sum of the coefficients of all of Karls favorite polynomials. (A monic quadratic is a quadratic polynomial whose x^2 term has a coefficient of 1.)

Solution: Notice that all of Karl's favorite polynomials are of the form $f(x) = (x-17)(x-c)$ where c lies in the range $[0, 34]$. Remembering that $f(1)$ is the sum of the coefficients of $f(x)$, we wish to compute:

$$\sum_{c=0}^{34} -16(1-c) = \sum_{c=0}^{34} 16(c-1) = 16 \cdot (\sum_{c=0}^{34} c - \sum_{c=0}^{34} 1)$$

Remembering that the sum of the positive integers from 1 to n is $\frac{n(n+1)}{2}$, we get that this is equal to $16 \cdot (\frac{34 \cdot 35}{2} - 35) = 16 \cdot (560) = \boxed{8960}$ as desired.

14. For exactly two real values of b, b_1 and b_2, the line $y = bx - 17$ intersects the parabola $y = x^2 + 2x + 3$ at exactly one point. Compute $b_1^2 + b_2^2$.

A point (x, y) that satisfies both $y = bx - 17$ and $y = x^2 + 2x + 3$ must satisfy $(x^2+2x+3) - (bx-17) = x^2 + (2-b) \cdot x + 20 = 0$. In order for this to have exactly one solution, the discriminant of this quadratic must be 0. It follows that $(2-b)^2 - 80 = 0$, or $b^2 - 4b - 76 = 0$. If we let the roots of this quadratic be b_1 and b_2, then by Vieta's Formulas, $b_1 + b_2 = 4$ and $b_1 \cdot b_2 = -76$. Finally, notice that

$$b_1^2 + b_2^2 = (b_1 + b_2)^2 - 2b_1 b_2 = 4^2 - 2 \times (-76) = \boxed{168}$$

as desired.

15. Compute the minimum possible value of $(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$ for real values of x.

Solution 1: Expanding this expression, we wish to compute the minimum possible value of $5x^2-30x+55=0$. Noting that the minimum possible value of a quadratic ax^2+bx+c will occur when $x=-\frac{b}{2a}$, we get that the value of x which minimizes this quadratic is $-\frac{-30}{2\cdot 5}=3$. Plugging this into the original expression gives us $2^2+1^2+0^2+(-1)^2+(-2)^2=\boxed{10}$ as desired.

Solution 2: Noticing that this expression must be an upward opening parabola and that it has symmetry about the line $x=3$, we know that the minimum value will occur at $x=3$. Plugging this in, we get $2^2+1^2+0^2+(-1)^2+(-2)^2=\boxed{10}$ as desired.

3 Sources

1. 2014 Stanford Math Tournament General Problem 1
2. 2014 Stanford Math Tournament General Problem 12
3. 2014 Stanford Math Tournament Algebra Problem 1
4. 2014 Stanford Math Tournament Algebra Problem 2
5. 2014 Stanford Math Tournament Algebra Problem 3
6. 2013 Stanford Math Tournament General Problem 1
7. 2013 Stanford Math Tournament General Problem 9
8. 2013 Stanford Math Tournament General Problem 12
9. 2013 Stanford Math Tournament General Problem 18
10. 2013 Stanford Math Tournament General Problem 23
11. 2013 Stanford Math Tournament Algebra Problem 1
12. 2013 Stanford Math Tournament Algebra Problem 4
13. 2013 Stanford Math Tournament Algebra Problem 3
14. 2013 Stanford Math Tournament Algebra Problem 5
15. 2012 Stanford Math Tournament Algebra Problem 1