Algebra Handout #4 Answers and Solutions

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December 19, 2017

1 Answers

1. -34

2. 4

3. 361

4. 10

5. $7 \cdot 3^{2014}$

6. 21

7. 22

8. $0, \log_2 3$

9. $\frac{7-\sqrt{17}}{2}$

10. $\frac{19}{12}$

11. 289

12. 252

13. 1000000

14. 3627

15. 3

2 Solutions

1. Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point.

Solution: Notice that the x-coordinates of the intersection points must be the solutions to the equation $x^3 + x + 2014 = ax + b$ for some constants a and b. By Vieta's Equations, the sum of these coordinates must be 0. It follows that our answer is $0 - 20 - 14 = \boxed{-34}$

2. Usually, spaceships have 6 wheels. However, there are more advanced spaceships that have 9 wheels. Aliens invade Earth with normal spaceships, advanced spaceships, and, surprisingly, bicycles (which have 2 wheels). There are 10 vehicles and 49 wheels in total. How many bicycles are there?

Solution: Let s be the number of spaceships, let a be the number of advanced spaceships, and let

b be the number of bicycles. It follows that $s + a + b = 10$ and $6s + 9a + 2b = 49$. Subtracting the first equation twice from the second equation gives us $4s + 7a = 29$. It follows that the only solution where $0 \le s, a$ and $s + a \le 10$ is $s = 2$, $a = 4$. It follows that $b = 10 - 2 - 4 = \boxed{4}$.

3. There is a square configuration of desks. It is known that one can rearrange these desks such that it has 7 fewer rows but 10 more columns, with 13 desks remaining. How many desks are there in the square configuration?

Solution: Let the number of rows in the square configuration be s. It follows that $s^2 = (s-7)(s+10) + 13 = s^2 + 3s - 57$. It follows that $3s = 57$, and therefore $s = 19$. It follows that our answer is $19^2 = \boxed{361}$.

4. Suppose three boba drinks and four burgers cost 28 dollars, while two boba drinks and six burgers cost \$37.70. If you paid for one boba drink using only pennies, nickels, dimes, and quarters, determine the least number of coins you could use.

Solution: Let the cost of a boba drink be o and let the cost of a burger be b. It follows that $3o+4b = 28$ and $2o+6b = 37.70$. Subtracting double the second equation from three times the first equation gives us $5o = 8.60$. It follows that $o = 1.72$. The minimum number of coins to express this value is $\boxed{10}$: we can use 6 quarters, 2 dimes, and 2 pennies.

5. Suppose $a_1, a_2, ...$ and $b_1, b_2, ...$ are sequences satisfying $a_n + b_n = 7, a_n = 2b_{n-1} - a_{n-1}$, and $b_n = 2a_{n-1} - b_{n-1}$, for all n. If $a_1 = 2$, find $(a_{2014})^2 - (b_{2014})^2$.

Solution: We can notice that $a_n + b_n = 2b_{n-1} - a_{n-1} + 2a_{n-1} - b_{n-1} = a_{n-1} + b_{n-1}$. It follows that $a_{2014} + b_{2014} = a_1 + b_1 = 7$. (I'm guessing they meant to say $a_1 + b_1 = 7$, but they actually already gave us this information in the original problem). We can also notice that $a_n - b_n = 3(b_{n-1} - a_{n-1}) = -3(a_{n-1} - b_{n-1})$. It follows that $a_{2014} - b_{2014} = (-3)^{2013}(a_1 - b_1) = 3^{2014}$. Therefore our answer is $(a_{2014} + b_{2014}) \cdot (a_{2014} - b_{2014}) = \boxed{7 \cdot 3^{2014}}$.

6. Suppose that $f(x) = \dfrac{x}{x^2 - 2x + 2}$ and $g(x_1, x_2, ..., x_7) = f(x_1) + f(x_2) + \cdots + f(x_7)$. If $x_1, x_2, ..., x_7$ are non-negative real numbers with sum 5, determine for how many tuples $(x_1, x_2, ..., x_7)$ does $g(x_1, x_2, ..., x_7)$ obtain its maximal value.

Solution: We will proceed with the infamous Tangent-Line Trick. Given that $f(x)$ can be rewritten as $f(x) = \frac{x}{(x-1)^2+1}$, we may be inclined to take the derivative of $f(x)$ at $x = 1$. Doing so gives us $\frac{d}{dx} = \frac{2-x^2}{(x^2-2x+2)^2}$, and it follows that the derivative at $x = 1$ is 1. It follows that the tangent line to $f(x)$ at $x = 1$ is $y = x$, and this makes us suspect that $f(x) \le x$. We can check this guess by rearranging the inequality to give us $x^3 - 2x^2 + x \ge 0$, or $x(x-1)^2 \ge 0$, which is true for all nonnegative values x with equality when $x = 0$ or $x = 1$. It follows that the 7-tuples which maximize $g(x_1, ..., x_7)$ consist of 5 1's and 2 0's. It follows that the number of 7-tuples which maximize this sum is $\dbinom{7}{2} = \boxed{21}$.

7. Let n be the smallest positive integers such that the number obtained by taking ns rightmost digit (decimal expansion) and moving it to be the leftmost digit is 7 times n. Determine the number of digits in n.

Solution: Let $n = 10x+a$ where $10^b \le x < 10^{b+1}$ and $0 \le a < 10$. It follows that $7n = 10^{b+1} \cdot a + x$. Therefore, $70x + 7a = 10^{b+1}a + x \to 69x = (10^{b+1} - 7)a$. Notice that if $a = 3$, then this equation can hold whenever $10^{b+1} \equiv 7 \pmod{23}$. The smallest value of b which satisfies this property is $b = 20$, as $10^{-1} \equiv 7 \pmod{23}$ and $10^{11} \equiv 22 \pmod{23}$ and by Euler's Totient Theorem, $10^{22} \equiv 1$

(mod 23). Therefore, $b = 20$, and n has $b + 2 = \boxed{22}$ digits.

8. Find all real numbers x such that $4^x - 2^{x+2} + 3 = 0$.

Solution: Let $2^x = a$. It follows that $a^2 - 4a + 3 = 0$, or $a = 1$ or $a = 3$. Therefore, the only solutions for x are $x = \boxed{0, \log_2 3}$

9. Find the smallest positive value of x such that $x^3 - 9x^2 + 22x - 16 = 0$.

Solution: We can notice that if the cubic is $f(x)$, $f(2) = 0$. Therefore, $(x - 2)$ is a factor of $f(x)$ and $f(x)$ can be written as $f(x) = (x - 2)(x^2 - 7x + 8)$. By the Quadratic Formula, the solutions to the quadratic are $\frac{7 \pm \sqrt{17}}{2}$. We can check that the minimum solution is therefore $\boxed{\dfrac{7 - \sqrt{17}}{2}}$.

10. Find $f(2)$ given that f is a real-valued function that satisfies the equation

$$4f(x) + (\frac{2}{3})(x^2 + 2)f(x - \frac{2}{x}) = x^3 + 1.$$

Solution: Notice that by plugging $x = 1$ and $x = -1$ into the given equation, we get $4f(1) + 2f(-1) = 2$ and $4f(-1) + 2f(1) = 0$. It follows that $2f(1) = \frac{4}{3}$, or $f(1) = \frac{2}{3}$. Plugging $x = 2$ into the original equation gives us $4f(2) + 4f(1) = 9$. It follows that $4f(2) = \frac{19}{3}$, and therefore $f(2) = \boxed{\dfrac{19}{12}}$.

11. Let $f(x) = x^2 + 18$ have roots r_1 and r_2, and let $g(x) = x^2 - 8x + 17$ have roots r_3 and r_4. If $h(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots $r_1 + r_3, r_1 + r_4, r_2 + r_3$, and $r_2 + r_4$, then find $h(4)$.

Solution: We wish to calculate $(4-r_1-r_3)(4-r_1-r_4)(4-r_2-r_3)(4-r_2-r_4)$. We can notice that r_1 and r_2 are in some order $3i\sqrt{2}, -3i\sqrt{2}$ and r_3 and r_4 are in some order $4+i, 4-i$. It follows that the expression we wish to find is equivalent to $(i+3i\sqrt{2})\cdot(i-3i\sqrt{2})(-i+3i\sqrt{2})(-i-3i\sqrt{2}) = 17^2 = \boxed{289}$.

12. Suppose that $x^{10} + x + 1 = 0$ and $x^{100} = a_0 + a_1x + ... + a_9x^9$. Find a_5.

Solution: Notice that $x^{100} = (x^{10})^{10} = (x + 1)^{10}$. It follows that the given 9th degree expression is $(x + 1)^{10} - x^{10} - x - 1$. The coefficient of the term with a degree of 5 in the expansion of this expression is $\binom{10}{5} = \boxed{252}$.

13. Let x be a real number such that $10^{\frac{1}{x}} = x$. Find $(x^3)^{2x}$.

Solution: We wish to compute $x^{6x} = (10^{\frac{1}{x}})^{6x} = 10^6 = \boxed{1000000}$

14. Let f be a function such that $f(x + y) = f(x) + f(y)$ for all x and y. Assume $f(5) = 9$. Compute $f(2015)$.

Solution: Notice that $f(5a) = f(5) + f(5a - 5)$. It follows that $f(5a) = 9a$. Therefore our answer is $\frac{2015 \cdot 9}{5} = \boxed{3627}$

15. Let x, y be non-zero solutions to $x^2 + xy + y^2 = 0$. Find

$$\frac{x^{2016} + (xy)^{1008} + y^{2016}}{(x + y)^{2016}}$$

Solution: Notice that if $x^2 + xy + y^2 = 0$, then $(x - y)(x^2 + xy + y^2) = x^3 - y^3 = 0$. It follows that $x^3 = y^3$. Therefore the given expression can be rewritten as $3(\frac{x}{x+y})^{2016}$. We can find that

$x = y(\frac{-1+\sqrt{3}}{2})$. It follows that $x + y = y(\frac{1+\sqrt{3}}{2})$. It follows that x has the same absolute value as $x + y$, and the difference between their arguments is $\frac{\pi}{3}$. Because 2016 is a multiple of 6, it follows that $3(\frac{x}{x+y})^{2016} = \boxed{3}$.

3 Sources

1. 2014 Berkeley Math Tournament Fall Individual Problem 16
2. 2014 Berkeley Math Tournament Fall Individual Problem 12
3. 2014 Berkeley Math Tournament Fall Team Problem 11
4. 2014 Berkeley Math Tournament Spring Individual Problem 3
5. 2014 Berkeley Math Tournament Spring Individual Problem 9
6. 2014 Berkeley Math Tournament Spring Individual Problem 14
7. 2014 Berkeley Math Tournament Spring Individual Problem 16
8. 2014 Berkeley Math Tournament Spring Analysis Problem 1
9. 2014 Berkeley Math Tournament Spring Analysis Problem 2
10. 2014 Berkeley Math Tournament Spring Analysis Problem 6
11. 2014 Berkeley Math Tournament Spring Analysis Problem 7
12. 2014 Berkeley Math Tournament Spring Team Problem 11
13. 2015 Berkeley Math Tournament Spring Individual Problem 4
14. 2015 Berkeley Math Tournament Spring Team Problem 1
15. 2015 Berkeley Math Tournament Fall Team Problem 13