Algebra Handout #4

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1 Problems

1. Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point.

2. Usually, spaceships have 6 wheels. However, there are more advanced spaceships that have 9 wheels. Aliens invade Earth with normal spaceships, advanced spaceships, and, surprisingly, bicycles (which have 2 wheels). There are 10 vehicles and 49 wheels in total. How many bicycles are there?

3. There is a square configuration of desks. It is known that one can rearrange these desks such that it has 7 fewer rows but 10 more columns, with 13 desks remaining. How many desks are there in the square configuration?

4. Suppose three boba drinks and four burgers cost 28 dollars, while two boba drinks and six burgers cost \$37.70. If you paid for one boba drink using only pennies, nickels, dimes, and quarters, determine the least number of coins you could use.

5. Suppose $a_1, a_2, ...$ and $b_1, b_2, ...$ are sequences satisfying $a_n + b_n = 7, a_n = 2b_{n-1} - a_{n-1}$, and $b_n = 2a_{n-1} - b_{n-1}$, for all n. If $a_1 = 2$, find $(a_{2014})^2 - (b_{2014})^2$.

6. Suppose that $f(x) = \dfrac{x}{x^2 - 2x + 2}$ and $g(x_1, x_2, ..., x_7) = f(x_1) + f(x_2) + \cdot\cdot + f(x_7)$. If $x_1, x_2, ..., x_7$ are non-negative real numbers with sum 5, determine for how many tuples $(x_1, x_2, ..., x_7)$ does $g(x_1, x_2, ..., x_7)$ obtain its maximal value.

7. Let n be the smallest positive integers such that the number obtained by taking ns rightmost digit (decimal expansion) and moving it to be the leftmost digit is 7 times n. Determine the number of digits in n.

8. Find all real numbers x such that $4^x - 2^{x+2} + 3 = 0$.

9. Find the smallest positive value of x such that $x^3 - 9x^2 + 22x - 16 = 0$.

10. Find $f(2)$ given that f is a real-valued function that satisfies the equation

$$4f(x) + (\frac{2}{3})(x^2 + 2)f(x - \frac{2}{x}) = x^3 + 1.$$

11. Let $f(x) = x^2 + 18$ have roots r_1 and r_2, and let $g(x) = x^2 - 8x + 17$ have roots r_3 and r_4. If $h(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots $r_1 + r_3, r_1 + r_4, r_2 + r_3$, and $r_2 + r_4$, then find $h(4)$.

12. Suppose that $x_{10} + x + 1 = 0$ and $x^{100} = a_0 + a_1x + ... + a_9x^9$. Find a_5.

13. Let x be a real number such that $10^{\frac{1}{x}} = x$. Find $(x^3)^{2x}$.

14. Let f be a function such that $f(x + y) = f(x) + f(y)$ for all x and y. Assume $f(5) = 9$. Compute $f(2015)$.

15. Let x, y be non-zero solutions to $x^2 + xy + y^2 = 0$. Find

$$\frac{x^{2016} + (xy)^{1008} + y^{2016}}{(x+y)^{2016}}$$

2 Sources

1. 2014 Berkeley Math Tournament Fall Individual Problem 16
2. 2014 Berkeley Math Tournament Fall Individual Problem 12
3. 2014 Berkeley Math Tournament Fall Team Problem 11
4. 2014 Berkeley Math Tournament Spring Individual Problem 3
5. 2014 Berkeley Math Tournament Spring Individual Problem 9
6. 2014 Berkeley Math Tournament Spring Individual Problem 14
7. 2014 Berkeley Math Tournament Spring Individual Problem 16
8. 2014 Berkeley Math Tournament Spring Analysis Problem 1
9. 2014 Berkeley Math Tournament Spring Analysis Problem 2
10. 2014 Berkeley Math Tournament Spring Analysis Problem 6
11. 2014 Berkeley Math Tournament Spring Analysis Problem 7
12. 2014 Berkeley Math Tournament Spring Team Problem 11
13. 2015 Berkeley Math Tournament Spring Individual Problem 4
14. 2015 Berkeley Math Tournament Spring Team Problem 1
15. 2015 Berkeley Math Tournament Spring Team Problem 13