Algebra Handout #5 Answers and Solutions

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1 Answers

1. $2,033,136$

2. $(5,-2,1)$

3. 3

4. 8

5. 2

6. -118

7. 552

8. 2

9. $\sqrt[3]{9}-2$

10. $-\frac{1}{2}, \frac{1}{2}\frac{3}{2}$

11. $\frac{20}{7}$

12. $(16,2)$

13. -1

14. 16

15. 73

2 Solutions

1. Let $g(x)=1+2x+3x^2+4x^3+....$ Find the coefficient of x^{2015} of $f(x)=\dfrac{g(x)}{1-x}$.

Solution: Notice that $g(x)$ is equivalent to the expansion of $(1+x+\cdots)^2=(\dfrac{1}{1-x})^2$. It follows that $f(x)=(\frac{1}{1-x})^3$. Therefore, we wish to find the number of triples of nonnegative integers (a,b,c) such that $a+b+c=2015$. By Stars and Bars, this is just $\dbinom{2017}{2}=\boxed{2,033,136}$.

2. Find all integer solutions to

$$x^2+2y^2+3z^2=36,$$
$$3x^2+2y^2+z^2=84,$$

$$xy + xz + yz = -7.$$

Solution: Adding the first two equations and dividing by 4, we get $x^2 + y^2 + z^2 = 30$. Combining this with the third equation, we get $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = (x + y + z)^2 = 16$. It follows that $x + y + z = 4$. Subtracting the first equation from the second equation, we get $x^2 - z^2 = 24$. Therefore, $x + z$ is an integer factor of 24, and because 24 is even, we must also have $x + z$ is even. Therefore, $x + z$ is among the set $\{-2, -4, -6, -8, -12, -24, 2, 4, 6, 8, 12, 24\}$. We also know that $xz + y(x + z) = -7$. Because $x + z$ is even, we know xz is odd, and therefore both x and z are odd. Just using the facts that $x + z$ is an even factor of 24, $x + y + z = 4$, and $xy + y(x + z) = -7$, we can find that the only value of $x + z$ which works $x + z = 6$. This produces the two triples $(x, y, z) = (1, -2, 5)$ and $(5, -2, 1)$. Checking, we find that only $(x, y, z) = \boxed{(5, -2, 1)}$ works.

3. Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1, a_2 = 2$ and for all $n \geq 3$,

$$a_{n+1} - a_n - a_{n+2} = 0.$$

Find $a_1 + a_2 + a_3 + ... + a_{2015}$.

Solution: Rewriting the recurrence as $a_{n+2} = a_{n+1} - a_n$, we can manually find that $a_3 = 3$, $a_4 = 1$, $a_5 = -2$, $a_6 = -3$, $a_7 = -1$, and $a_8 = 2$, with the sequence repeating every six terms. Notice that every block of 6 consecutive terms adds to 0, so our answer is $a_1 + a_2 + a_3 + a_4 + a_5 = -1 + 2 + 3 + 1 - 2 = \boxed{3}$.

4. Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

Solution: Notice that the given equation is equivalent to $(x - 2)^2 + y^2 = 1$, an equation which produces a circle of radius 1 centered at $(2, 0)$. It follows that the maximum value of $x^2 + y^2$ corresponds to the point on this circle which is furthest from the origin, or $(3, 0)$, and therefore $M = 9$. Similarly, m corresponds to the point $(1, 0)$ and therefore $m = 1$. It follows that $M - m = \boxed{8}$.

5. Given integers a, b, c satisfying

$$abc + a + c = 12$$

$$bc + ac = 8$$

$$b - ac = -2,$$

what is the value of a?

Solution: Adding the last two equations, we get $b(c + 1) = 6$. Therefore, $c + 1$ is a factor of 6. By the second equation we know that c is a factor of 8, so it follows that $c = -4, -2, 1$ or 2. Plugging in each of these values of c and solving tells us that the only solution (a, b, c) in integers is $(a, b, c) = (2, 2, 2)$, and therefore $a = \boxed{2}$.

6. Consider the following linear system of equations.

$$1 + a + b + c + d = 1$$

$$16 + 8a + 4b + 2c + d = 2$$

$$81 + 27a + 9b + 3c + d = 3$$

,

$$256 + 64a + 16b + 4c + d = 4$$

Find $a-b+c-d$.

Solution: Consider the polynomial $f(x)=x^4+ax^3+bx^2+cx+d$. The given equations tell us that for $1\le x\le 4$, $f(x)=x$. Therefore, $f(x)=(x-1)(x-2)(x-3)(x-4)+x$. Notice that $f(-1)=1-a+b-c+d=119$. Therefore $a-b+c-d=\boxed{-118}$.

7. Positive integers x,y,z satisfy $(x+yi)^2-46i=z$. What is $x+y+z$?

Solution: The given equation tells us that $2xy=46$, or $xy=23$ and that $x^2-y^2=z$. Because z is positive, we must have $x>y$, and therefore $x=23$ and $y=1$ and $z=528$. It follows that $x+y+z=529+23=\boxed{552}$.

8. Define $P(\tau)=(\tau+1)^3$. If $x+y=0$, what is the minimum possible value of $P(x)+P(y)$?

Solution: We wish to find the minimum possible value of $(1+x)^3+(1-x)^3$. This is equivalent to the minimum possible value of $6x^2+2$ which is clearly $\boxed{2}$ at $x=0$.

9. Simplify $\dfrac{1}{\sqrt[3]{81}+\sqrt[3]{72}+\sqrt[3]{64}}$

Solution: The given expression is equivalent to $\dfrac{1}{4+2\sqrt[3]{9}+\sqrt[3]{9^2}}$. If we let $a=2$ and $b=\sqrt[3]{9}$, then the given expression is equivalent to $\frac{1}{a^2+ab+b^2}=\frac{a-b}{a^3-b^3}$. Substituting our values of a and b into this new expression gives us an answer of $\dfrac{\sqrt[3]{9}-2}{1}=\boxed{\sqrt[3]{9}-2}$.

10. The roots of the polynomial $x^3-\dfrac{3}{2}x^2-\dfrac{1}{4}x+\dfrac{3}{8}=0$ are in arithmetic progression. What are they?

Solution: Let the roots be a,b, and c where $a<b<c$, and let their common difference be d such that $a+d=b$ and $b+d=c$. It follows that $a+b+c=(b-d)+b+(b+d)=3b=\frac{3}{2}$ by Vieta's. Therefore $b=\frac{1}{2}$ is a root. Factoring this out, we get that the polynomial is equivalent to $(x-\frac{1}{2})(x^2-x-\frac{3}{4})$. Using the Quadratic Formula on the second factor, we get that $x=\frac{3}{2}$ and $x=-\frac{1}{2}$ are also roots. Therefore our answer is $\boxed{-\dfrac{1}{2},\dfrac{1}{2},\dfrac{3}{2}}$.

11. The quartic equation $x^4+2x^3-20x^2+8x+64$ contains the points $(-6,160),(-3,-113)$, and $(2,32)$. A cubic $y=ax^3+bx+c$ also contains these points. Determine the x-coordinate of the fourth intersection of the cubic with the quartic.

Solution: The given information is equivalent to the expression $x^4+(2-a)x^3-20x^2+(8-b)x+64-c=(x+6)(x+3)(x-2)(x-d)=0$ where d is the x-coordinate of the fourth intersection point. Equating their x^2 terms gives us $-20=18-6-12-6d-3d+2d$ or $-20=-7d$. It follows that the fourth x-coordinate is $\boxed{\dfrac{20}{7}}$.

12. Find an integer pair of solutions (x,y) to the following system of equations.

$$\log_2(y^x)=16$$
$$\log_2(x^y)=8$$

Solution: Notice that the given equations are equivalent to $x\log_2 y=16$ and $y\log_2 x=8$. If we let $x=2^a$ and $y=2^b$, then it follows that $b\cdot 2^a=16$ and $a\cdot 2^b=8$. We can notice that the pair $(a,b)=(4,1)$ satisfies this equation, and therefore the pair $(x,y)=\boxed{(16,2)}$ satisfies the original equation.

13. Define a_n such that $a_1 = \sqrt{3}$ and for all integers i, $a_{i+1} = a_i^2 - 2$. What is a_{2016}?

Solution: Notice that $a_2 = 1$, and $a_3 = -1$. Notice that no matter how many times the function $f(x) = x^2 - 2$ is applied to $x = -1$, we will always have $f(-1) = -1$, and therefore for all $n \geq 3$, $a_n = -1$. It follows that $a_{2016} = \boxed{-1}$.

14. Let s_1, s_2, s_3 be the three roots of $x^3 + x^2 + \dfrac{9}{2}x + 9$.

$$\prod_{i=1}^{3}(4s_i^4 + 81)$$

can be written as $2^a 3^b 5^c$. Find $a + b + c$.

Solution: Consider the cubic $x^3 - dx^2 + ex - f$ with roots g, h, and i. By Vieta's Formulas, we know $g + h + i = d$, $gh + gi + hi = e$, and $ghi = f$. It follows that $g^2 + h^2 + i^2 = d^2 - 2e$, $g^2h^2 + g^2i^2 + h^2i^2 = e^2 - 2df$, and $g^2h^2i^2 = f^2$. It follows that the cubic with roots g^2, h^2, and i^2 is $x^3 - (d^2 - 2e)x^2 + (e^2 - 2df)x - f^2$. Using this knowledge, we can find that the cubic with roots s_1^2, s_2^2, and s_3^2 is $x^3 + 8x^2 + \frac{9}{4}x - 81$ and the cubic with roots s_1^4, s_2^4, and s_3^4 is $f(x) = x^3 - \frac{119}{2}x^2 + \frac{20817}{16}x - 6561$. Notice that the expression we want is equivalent to $-64(-\frac{81}{4} - s_1^4)(-\frac{81}{4} - s_2^4)(-\frac{81}{4} - s_2^4) = -64f(-\frac{81}{4})$. It follows that the given product is equal to $4199040 = 2^7 3^8 5^1$. It follows that $a + b + c = 7 + 8 + 1 = \boxed{16}$.

15. $(\sqrt{6} + \sqrt{7})^{1000}$ in base ten has a tens digit of a and a ones digit of b. Determine $10a + b$.

Solution: Notice that by adding $(\sqrt{7} - \sqrt{6})^{1000}$ to $(\sqrt{6} + \sqrt{7})^{1000}$ and using the Binomial Theorem on both powers, all of the radical terms of the expansions will cancel each other out, leaving us with the following integer:

$$(\sqrt{6} + \sqrt{7})^{1000} + (\sqrt{7} - \sqrt{6})^{1000} = 2 \cdot 7^{500} + 2 \cdot \binom{1000}{2} \cdot 7^{499} \cdot 6 + \cdots + 2 \cdot \binom{1000}{998} \cdot 6^{499} \cdot 7^1 + 2 \cdot 6^{500}$$

Let $\sqrt{6} + \sqrt{7} = a$, and let $\sqrt{7} - \sqrt{6} = b$. It follows that we wish to calculate $a^{1000} + b^{1000} \pmod{100}$. Let $F_n = a^{2n} + b^{2n}$. We know that $ab = 1$ by difference of squares and we can manually calculate that $a^2 + b^2 = 26$. Notice that $a^{2n+2} + b^{2n+2} = (a^2 + b^2)(a^{2n} + b^{2n}) - a^2b^2(a^{2n-2} + b^{2n-2})$, or in other words, $F_{n+1} = 26F_n - F_{n-1}$. We can also calculate that $F_1 = 26$ and $F_2 \equiv 74 \pmod{100}$. We wish to find $F_{500} - 1 \pmod{100}$ as $b^{1000} < 1$. We can do this by finding $F_{500} \pmod{25}$ and $F_{500} \pmod{4}$. Let $G_n = F_n \pmod{25}$. It follows that $G_1 = 1$ and $G_2 = 24$ and that $G_n = (G_{n-1} - G_{n-2}) \pmod{25}$. From here we can manually find that $G_3 = 23$, $G_4 = 24$, $G_5 = 1$, $G_6 = 2$, $G_7 = 1$, and $G_8 = 24$, therefore creating a cycle of length 6. It follows that $G_{500} = G_2 = 24$, and therefore $F_{500} \equiv 24 \pmod{25}$. Let $H_n = F_n \pmod{4}$. We know that $H_1 = H_2 = 2$, and $H_n = (-H_{n-2}) \pmod{4}$. It follows that $H_n = 2$ for all n, and therefore $H_{500} \equiv F_{500} \equiv 2 \pmod{4}$. By the Chinese Remainder Theorem, it follows that $F_{500} \equiv 74 \pmod{100}$, and therefore our answer is $74 - 1 = \boxed{73}$.

3 Sources

1. 2015 Berkeley Math Tournament Spring Analysis Problem 2
2. 2015 Berkeley Math Tournament Spring Analysis Problem 3
3. 2015 Berkeley Math Tournament Spring Analysis Problem 4
4. 2015 Berkeley Math Tournament Spring Analysis Problem 5

5. 2015 Berkeley Math Tournament Fall Individual Problem 15
6. 2015 Berkeley Math Tournament Fall Individual Problem 19
7. 2016 Berkeley Math Tournament Spring Individual Problem 5
8. 2016 Berkeley Math Tournament Spring Individual Problem 7
9. 2016 Berkeley Math Tournament Spring Individual Problem 8
10. 2016 Berkeley Math Tournament Spring Individual Problem 11
11. 2016 Berkeley Math Tournament Spring Individual Problem 13
12. 2016 Berkeley Math Tournament Spring Analysis Problem 2
13. 2016 Berkeley Math Tournament Spring Team Problem 1
14. 2016 Berkeley Math Tournament Spring Team Problem 15
15. 2016 Berkeley Math Tournament Spring Discrete Problem 9