Algebra Handout #5

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1 Problems

1. Let $g(x) = 1 + 2x + 3x^2 + 4x^3 + ...$ Find the coefficient of x^{2015} of $f(x) = \frac{g(x)}{1-x}$.

2. Find all integer solutions to

$$x^2 + 2y^2 + 3z^2 = 36,$$
$$3x^2 + 2y^2 + z^2 = 84,$$
$$xy + xz + yz = -7.$$

3. Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1, a_2 = 2$ and for all $n \geq 3$,

$$a_{n+1} - a_n - a_{n+2} = 0.$$

Find $a_1 + a_2 + a_3 + ... + a_{2015}$.

4. Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

5. Given integers a, b, c satisfying

$$abc + a + c = 12$$
$$bc + ac = 8$$
$$b - ac = -2,$$

what is the value of a?

6. Consider the following linear system of equations.

$$1 + a + b + c + d = 1$$
$$16 + 8a + 4b + 2c + d = 2$$
$$81 + 27a + 9b + 3c + d = 3$$

,

$$256 + 64a + 16b + 4c + d = 4$$

Find $a - b + c - d$.

7. Positive integers x, y, z satisfy $(x + yi)^2 - 46i = z$. What is $x + y + z$?

8. Define $P(\tau) = (\tau + 1)^3$. If $x + y = 0$, what is the minimum possible value of $P(x) + P(y)$?

9. Simplify $\frac{1}{\sqrt[3]{81} + \sqrt[3]{72} + \sqrt[3]{64}}$

10. The roots of the polynomial $x^3 - \frac{3}{2}x^2 - \frac{1}{4}x + \frac{3}{8} = 0$ are in arithmetic progression. What are they?

11. The quartic equation $x^4 + 2x^3 - 20x^2 + 8x + 64$ contains the points $(-6, 160), (-3, -113)$, and $(2, 32)$. A cubic $y = ax^3 + bx + c$ also contains these points. Determine the x-coordinate of the fourth intersection of the cubic with the quartic.

12. Find an integer pair of solutions (x, y) to the following system of equations.

$$\log_2(y^x) = 16$$

$$\log_2(x^y) = 8$$

13. Define a_n such that $a_1 = \sqrt{3}$ and for all integers i, $a_{i+1} = a_i^2 - 2$. What is a_{2016}?

14. Let s_1, s_2, s_3 be the three roots of $x^3 + x^2 + \frac{9}{2}x + 9$.

$$\prod_{i=1}^{3}(4s_i^4 + 81)$$

can be written as $2^a 3^b 5^c$. Find $a + b + c$.

15. $(\sqrt{6} + \sqrt{7})^{1000}$ in base ten has a tens digit of a and a ones digit of b. Determine $10a + b$.

2 Sources

1. 2015 Berkeley Math Tournament Spring Analysis Problem 2
2. 2015 Berkeley Math Tournament Spring Analysis Problem 3
3. 2015 Berkeley Math Tournament Spring Analysis Problem 4
4. 2015 Berkeley Math Tournament Spring Analysis Problem 5
5. 2015 Berkeley Math Tournament Fall Individual Problem 15
6. 2015 Berkeley Math Tournament Fall Individual Problem 19
7. 2016 Berkeley Math Tournament Spring Individual Problem 5
8. 2016 Berkeley Math Tournament Spring Individual Problem 7
9. 2016 Berkeley Math Tournament Spring Individual Problem 8
10. 2016 Berkeley Math Tournament Spring Individual Problem 11
11. 2016 Berkeley Math Tournament Spring Individual Problem 13
12. 2016 Berkeley Math Tournament Spring Analysis Problem 2
13. 2016 Berkeley Math Tournament Spring Team Problem 1
14. 2016 Berkeley Math Tournament Spring Team Problem 15
15. 2016 Berkeley Math Tournament Spring Discrete Problem 9