Algebra Handout #8 Answers and Solutions

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1 Answers

1. 265

2. 16

3. 2526

4. 9

5. 2

6. $\frac{12}{35}$

7. 46207

8. -95

9. $\frac{55}{8} = \$6.875$

10. $\frac{7}{4}$

11. 18

12. $(4, 1)$

13. 12

14. 384

15. -2

2 Solutions

1. One evening, Varun finishes reading a novel that he has been reading for several days and finds the ending so exciting that he immediately begins reading the novel's sequel. Each novel has pages numbered consecutively, starting with page 1. Each novel has fewer than 1000 pages. If Varun reads a total of 42 pages in one sitting and the sum of the page numbers he reads in that sitting is 2018, what is the number of the last page of the first novel?

Solution: Let the number of pages in the first novel be n, and let the number of pages that Varun read in the first novel be $p+1$. Then we have that $\frac{(2n-p)(p+1)}{2} + \frac{(41-p)(42-p)}{2} = 2018$. Rearranging, we get $np + n - 42p = 1157$. It follows that $(p+1)(n-42) = 1115$. We can notice that the only pair (p, n) which satisfies this relationship is $(p, n) = (4, 265)$. It follows that our answer is $\boxed{265}$.

2. Two lines with slopes m and n, with $m > n > 0$, intersect at the origin. The line $y = x$ bisects the angle between the two lines. If $m+n=2\sqrt{65}$, what is the value of $m-n$?

Solution: Define the *argument* of a line l that passes through the origin to be the angle that l makes with the positive x-axis. The given conditions tell us that if the line with slope m has *argument* α and the line with slope n has *argument* β, then $\alpha+\beta=2\cdot 45°=90°$. It follows that the angle that the line with slope n makes with the positive x-axis is the same as the angle that the line with slope m makes with the positive y-axis. Therefore, if we let $\tan\beta=\frac{a}{b}$, then we must have that $n=\frac{a}{b}$ and $m=\frac{b}{a}$. It follows that $m=\frac{1}{n}$.

It follows that we know $m+\frac{1}{m}=2\sqrt{65}$. Squaring both sides, we get $m^2+\frac{1}{m^2}=258$. If we let $m^2=a$, then it follows that $a^2-258a+1=0$. The roots of this polynomial are consequently m^2 and n^2. It follows that $m^2-n^2=\sqrt{258^2-4}=2\sqrt{128\cdot 130}=32\sqrt{65}$. Therefore, we have that:

$$m-n=\frac{m^2-n^2}{m+n}=\frac{32\sqrt{65}}{2\sqrt{65}}=\boxed{16}$$

3. If a certain sequence $a_1, a_2, a_3, a_4, ...$ of positive integers has the following properties, what is the greatest possible value of a_{99}?

For every positive integer $k, a_k < a_{k+1}$.
For every positive integer $k>3$, $a_{k-3}+a_{k-2}+a_{k-1}+a_k=k^2$.

Solution: By the given properties, we know that in general, $a_k-a_{k-4}=k^2-(k-1)^2=2k-1$. It follows that $a_{99}=a_3+13+21+29+\cdots+197=a_3+210\cdot\frac{24}{2}=a_3+2520$. It follows that the greatest possible value of a_{99} corresponds to the greatest possible value of a_3. We can easily find that this is equivalent to 6, when $a_1=1$, $a_2=2$, $a_3=6$, and $a_4=7$. Therefore, our answer is $2520+6=\boxed{2526}$.

4. If A and B represent digits such that $\dfrac{A}{B}=\dfrac{A6}{6B}=\dfrac{A66}{66B}=\dfrac{A666}{666B}=\dfrac{A6666}{6666B}$, what is the sum of all possible digits A?

Solution: Consider the first two fractions in this equation. The fact that they are equal tells us that $B(10A+6)=A(60+B)$ or $3AB=20A-2B$. Using Simon's Favorite Factoring Trick, we can find that this is equivalent to $(3A+2)(3B-20)=-40$. It follows that solutions for A and B correspond to factor pairs of -40. We must have that $B>0$, so it follows that $3B-20\geq -17$. It follows that the only possible values of $3B-20$ are -8, -5, and -2. These correspond to the solutions $(A,B)=(1,4),(2,5)$, and $(6,6)$. Plugging these into the other fractions shows that each of these pairs of digits works. Therefore, the sum of the possible values of A is $1+2+6=\boxed{9}$.

5. Real numbers m and n exist such that $(n+2)^2-(n-2)^2=(m+1)^2-(m-1)^2$. If m and n are nonzero and $m=an$, what is the value of a?

Solution: Rearranging the given equation tells us that $8n=4m$, or $2n=m$. It follows that $a=\boxed{2}$.

6. There are two values of x such that $\dfrac{|x-2018|}{|x-2019|}=\dfrac{1}{6}$. What is the absolute difference between these two values of x? Express your answer as a common fraction.

Solution: We have three cases. Either $x>2019$, $2018\leq x<2019$, or $x<2018$. In the first case, we know that $x-2019=6x-12108$. Rearranging this tells us that $x=\frac{10089}{5}$. However, this value of x is less than 2019, so we have no solutions in this case. In the second case, we know that $2019 - x = 6x - 12108$. Rearranging, we get $x = \frac{14127}{7}$, which does fall in the given range. In the third case, we know that $2019 - x = 12108 - 6x$, which tells us that $x = \frac{10089}{5}$. This falls in the given range, so we know that this value of x also works. It follows that our answer is $\frac{14127}{7} - \frac{10089}{5} = \boxed{\frac{12}{35}}$.

7. In the equation shown, if p and q are positive integers and p is odd, what is the value of the sum $p + q$?

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{15}{16} \cdot \frac{17}{18} \cdot \frac{19}{20} = \frac{p}{2^q}$$

Solution: We can notice that the numerator of this product is equivalent to $\frac{20!}{2^{10} \cdot 10!}$. Similarly, we can notice that the denominator of this product is equivalent to $2^{10} \cdot 10!$. It follows that this product is equivalent to $\frac{\binom{20}{10}}{2^{20}} = \frac{46189}{2^{18}}$. It follows that our answer is $18 + 46189 = \boxed{46207}$.

8. In the equation $y = 3 - \sqrt{\frac{4-x}{2}}$, what is the sum of all the integer values of x that make y nonnegative?

Solution: By the given conditions, we must have $x \leq 4$. In addition, we must also have that $\frac{4-x}{2} \leq 9$, or $x \geq -14$. It follows that we wish to add all of the numbers from -14 to 4. We can easily find that this is $(4 - 14) \cdot (4 + 14 + 1) \cdot \frac{1}{2} = \boxed{-95}$.

9. Noah is combining gummy bears and jelly beans in equal parts to create a mixture that he will sell. The bears cost him \$20 for 8 pounds, while the beans cost \$14 for 4 pounds. He wants his cost to be 40% of his selling price. At what price per pound should he sell the mixture?

Solution: Notice that Noah must pay $20 + 14 + 14 = \$44$ for 16 pounds of the mixture. It follows that his selling price should be $\frac{44}{16} \cdot \frac{5}{2} = \boxed{\frac{55}{8} = 6.875}$.

10. Let $f(x) = \frac{x - 3a}{x - 2b}$ for constants a and b. If $f(5) = 0$ and $f(3)$ is undefined, what is the value of $f(\frac{1}{3})$? Express your answer as a common fraction.

Solution: By the given conditions, we must have that $3 - 2b = 0$ and $5 - 3a = 0$. It follows that $f(x) = \frac{x-5}{x-3}$. Therefore, $f(\frac{1}{3}) = \frac{-\frac{14}{3}}{-\frac{8}{3}} = \boxed{\frac{7}{4}}$.

11. If a, b, c, d, and e are constants such that every $x > 0$ satisfies

$$\frac{5x^4 - 8x^3 + 2x^2 + 4x + 7}{(x+2)^4} = a + \frac{b}{x+2} + \frac{c}{(x+2)^2} + \frac{d}{(x+2)^3} + \frac{e}{(x+2)^4},$$

then what is the value of $a + b + c + d + e$?

Solution: If this equation is satisfied for ever $x > 0$, then it must be satisfied for all values of x which are not equal to -2. We can notice that when $x = -1$, the right hand side of this equation is $a + b + c + d + e$, and the left hand side of this equation is $\boxed{18}$.

12. The *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let $G_n = F_{3n}$ be every third Fibonacci number. There are constants a and b such that every integer $n \geq 2$ satisfies

$$G_n = aG_{n-1} + bG_{n-2}.$$

Compute the ordered pair (a, b).

Solution: Through brute force, we can calculate that $G_0 = 0$, $G_1 = 2$, $G_2 = 8$, and $G_3 = 34$. It follows that $2a = 8$ and $8a + 2b = 34$. It follows that $a = 4$ and $b = 1$. Therefore our answer is $\boxed{(4, 1)}$.

13. The three roots of the cubic $30x^3 - 50x^2 + 22x - 1$ are distinct real numbers between 0 and 1. For every nonnegative integer n, let s_n be the sum of the nth powers of these three roots. What is the value of the infinite series

$$s_0 + s_1 + s_2 + s_3 + ...?$$

Solution: If we let the roots be d, e, and f, then we wish to calculate

$$\frac{1}{1-d} + \frac{1}{1-e} + \frac{1}{1-f} = \frac{(1-d)(1-e) + (1-d)(1-f) + (1-e)(1-f)}{(1-d)(1-e)(1-f)} =$$

$$= \frac{3 + de + df + ef - 2d - 2e - 2f}{(1-d)(1-e)(1-f)}$$

By Vieta's Formulas, $de + df + ef = \frac{11}{15}$ and $d + e + f = \frac{5}{3}$. It follows that the numerator of the fraction we want is $3 + \frac{11}{15} + -\frac{10}{3} = \frac{2}{5}$. We know that the given cubic is equivalent to $g(x) = 30(x - d)(x - e)(x - f)$, so it follows that the denominator of the fraction we want is $\frac{g(1)}{30} = \frac{1}{30}$. It follows that our answer is $\frac{\frac{2}{5}}{\frac{1}{30}} = \boxed{12}$.

14. Compute the value of the expression

$$2009^4 - 4 \times 2007^4 + 6 \times 2005^4 - 4 \times 2003^4 + 2001^4$$

Solution: Notice that the expansion of $(x+4)^4 - 4(x+2)^4 + 6x^4 - 4(x-2)^4 + (x-4)^4$ is $512 - 128 = 384$. It follows that our answer is $\boxed{384}$.

15. Let a, b, c, x, y, and z be real numbers that satisfy the three equations

$$13x + by + cz = 0$$
$$ax + 23y + cz = 0$$
$$ax + by + 42z = 0.$$

Suppose that $a \neq 13$ and $x \neq 0$. What is the value of

$$\frac{13}{a-13} + \frac{23}{b-23} + \frac{42}{c-42}?$$

Solution: By subtracting each of the three equations from each other, we can get that $(23-b)y = (13-a)x = (42-c)z$. By adding these expressions to each of the three equations, we can find that $13x + 23y + 42z = 2x(13-a) = 2y(23-b) = 2z(42-c)$. It follows that $\frac{1}{a-13} = -\frac{2x}{13x+23y+42z}$, $\frac{1}{b-23} = -\frac{2y}{13x+23y+42z}$, and $\frac{1}{c-42} = -\frac{2z}{13x+23y+42z}$. It follows that the expression we want is equivalent to $\frac{-26x-46y-84z}{13x+23y+42z} = \boxed{-2}$.

3 Sources

1. 2018 Mathcounts State Target Round Problem 8
2. 2018 Mathcounts State Target Round Problem 6
3. 2018 Mathcounts State Team Round Problem 10
4. 2018 Mathcounts State Team Round Problem 6
5. 2018 Mathcounts State Team Round Problem 3
6. 2018 Mathcounts Chapter Sprint Round Problem 29
7. 2018 Mathcounts Chapter Sprint Round Problem 24
8. 2018 Mathcounts Chapter Target Round Problem 8
9. 2018 Mathcounts Chapter Team Round Problem 9
10. 2018 Mathcounts Chapter Team Round Problem 6
11. 2009 Math Prize For Girls Problem 2
12. 2009 Math Prize For Girls Problem 3
13. 2009 Math Prize For Girls Problem 14
14. 2009 Math Prize For Girls Problem 7
15. 2009 Math Prize For Girls Problem 17