

Algebra Handout #8

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1 Problems

1. One evening, Varun finishes reading a novel that he has been reading for several days and finds the ending so exciting that he immediately begins reading the novel's sequel. Each novel has pages numbered consecutively, starting with page 1. Each novel has fewer than 1000 pages. If Varun reads a total of 42 pages in one sitting and the sum of the page numbers he reads in that sitting is 2018, what is the number of the last page of the first novel?
2. Two lines with slopes m and n , with $m > n > 0$, intersect at the origin. The line $y = x$ bisects the angle between the two lines. If $m + n = 2\sqrt{65}$, what is the value of $m - n$?
3. If a certain sequence $a_1, a_2, a_3, a_4, \dots$ of positive integers has the following properties, what is the greatest possible value of a_{99} ?

For every positive integer k , $a_k < a_{k+1}$.

For every positive integer $k > 3$, $a_{k-3} + a_{k-2} + a_{k-1} + a_k = k^2$.

4. If A and B represent digits such that $\frac{A}{B} = \frac{A6}{6B} = \frac{A66}{66B} = \frac{A666}{666B} = \frac{A6666}{6666B}$, what is the sum of all possible digits A ?
5. Real numbers m and n exist such that $(n + 2)^2 - (n - 2)^2 = (m + 1)^2 - (m - 1)^2$. If m and n are nonzero and $m = an$, what is the value of a ?
6. There are two values of x such that $\frac{|x - 2018|}{|x - 2019|} = \frac{1}{6}$. What is the absolute difference between these two values of x ? Express your answer as a common fraction.
7. In the equation shown, if p and q are positive integers and p is odd, what is the value of the sum $p + q$?

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{15}{16} \cdot \frac{17}{18} \cdot \frac{19}{20} = \frac{p}{2^q}$$

8. In the equation $y = 3 - \sqrt{\frac{4-x}{2}}$, what is the sum of all the integer values of x that make y nonnegative?
9. Noah is combining gummy bears and jelly beans in equal parts to create a mixture that he will sell. The bears cost him \$20 for 8 pounds, while the beans cost \$14 for 4 pounds. He wants his cost to be 40% of his selling price. At what price per pound should he sell the mixture?

10. Let $f(x) = \frac{x - 3a}{x - 2b}$ for constants a and b . If $f(5) = 0$ and $f(3)$ is undefined, what is the value of $f(\frac{1}{3})$? Express your answer as a common fraction.

11. If $a, b, c, d,$ and e are constants such that every $x > 0$ satisfies

$$\frac{5x^4 - 8x^3 + 2x^2 + 4x + 7}{(x + 2)^4} = a + \frac{b}{x + 2} + \frac{c}{(x + 2)^2} + \frac{d}{(x + 2)^3} + \frac{e}{(x + 2)^4},$$

then what is the value of $a + b + c + d + e$?

12. The *Fibonacci numbers* are defined recursively by the equation

$$F_n = F_{n-1} + F_{n-2}$$

for every integer $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Let $G_n = F_{3n}$ be every third Fibonacci number. There are constants a and b such that every integer $n \geq 2$ satisfies

$$G_n = aG_{n-1} + bG_{n-2}.$$

Compute the ordered pair (a, b) .

13. The three roots of the cubic $30x^3 - 50x^2 + 22x - 1$ are distinct real numbers between 0 and 1. For every nonnegative integer n , let s_n be the sum of the n th powers of these three roots. What is the value of the infinite series

$$s_0 + s_1 + s_2 + s_3 + \dots?$$

14. Compute the value of the expression

$$2009^4 - 4 \times 2007^4 + 6 \times 2005^4 - 4 \times 2003^4 + 2001^4$$

15. Let $a, b, c, x, y,$ and z be real numbers that satisfy the three equations

$$13x + by + cz = 0$$

$$ax + 23y + cz = 0$$

$$ax + by + 42z = 0.$$

Suppose that $a \neq 13$ and $x \neq 0$. What is the value of

$$\frac{13}{a - 13} + \frac{23}{b - 23} + \frac{42}{c - 42}?$$

2 Sources

1. 2018 Mathcounts State Target Round Problem 8
2. 2018 Mathcounts State Target Round Problem 6
3. 2018 Mathcounts State Team Round Problem 10
4. 2018 Mathcounts State Team Round Problem 6
5. 2018 Mathcounts State Team Round Problem 3

6. 2018 Mathcounts Chapter Sprint Round Problem 29
7. 2018 Mathcounts Chapter Sprint Round Problem 24
8. 2018 Mathcounts Chapter Target Round Problem 8
9. 2018 Mathcounts Chapter Team Round Problem 9
10. 2018 Mathcounts Chapter Team Round Problem 6
11. 2009 Math Prize For Girls Problem 2
12. 2009 Math Prize For Girls Problem 3
13. 2009 Math Prize For Girls Problem 14
14. 2009 Math Prize For Girls Problem 7
15. 2009 Math Prize For Girls Problem 17