Triangles

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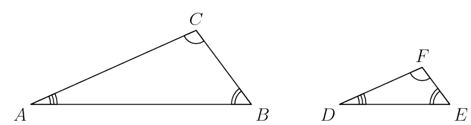
1 Similarity and Congruence

1.1 Similarity

Two triangles are similar if one can be turned, flipped, and/or dilated to exactly match the other. This means that everything but size and orientation are the same.

1.2 AAA Similarity



Theorem 1. *Two triangles are similar iff they are made up of the same angles.*

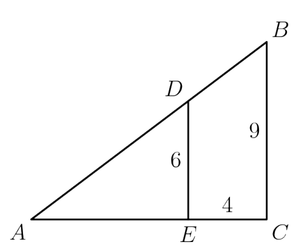
1.3 AA Similarity

Since two angles of a triangle define the third, two triangles can be proven similar if two pairs of angles are equal. The third pair of equal angles is unnecessary.

Theorem 2. *Two triangles are similar iff the ratios of corresponding sides are all equal. Therefore, if $\triangle ABC$ is similar to $\triangle DEF$,*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

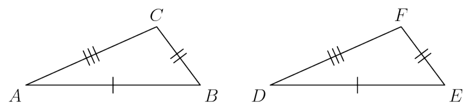
Example: In $\triangle ABC$, C is a right angle, D is on AB, E is on AC and $\angle AED$ is a right angle. If $BC = 9$ mm, $DE = 6$ mm and EC = 4 mm, what is AB?



1.4 Congruence

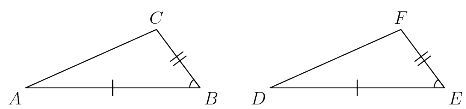
Two triangles are congruent if one of them can be turned and/or flipped and placed exactly on top of the other, with all parts lining up perfectly. Or more simply, two objects are congruent if they have the same size and shape.

1.5 SSS Congruence



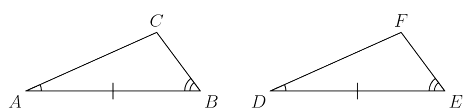
If three sides of one triangle are congruent to to the corresponding sides of another triangle, then the two triangles are congruent.

1.6 SAS congruence



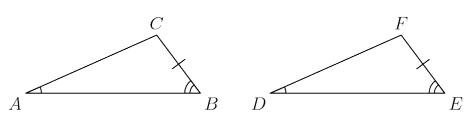
If two sides of two triangles are congruent, and the angle between the two sides is equal, then the two triangles are congruent.

1.7 ASA congruence



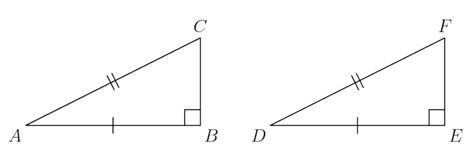
If two angle of two triangles are equal, and the side between the two angles is the same length, then the two triangles are congruent.

1.8 AAS congruence



If two angles of two triangles are equal, and corresponding sides of the triangle (not between the angles) are congruent, then the two triangles are congruent. This is an extension of ASA congruence, as given two triangles with two pairs of congruent angles, the last pair of angles is also congruent.

1.9 HL congruence



This is a special case of congruency involving right triangles. If both the hypotenuse and leg of one right triangle are congruent to that of another, then the two triangles are congruent.

We can use the knowledge of congruence to discover missing information about triangles. If two triangles are known to be congruent, then all corresponding parts of the triangles are also congruent.

2 Pythagorean Theorem

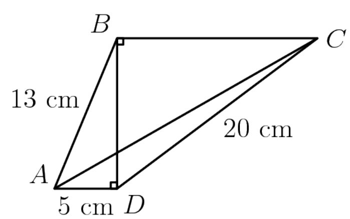
A right triangle is a special type of triangle where one of the angles is 90 degrees. (In other words, one of the sides is the altitude from one vertex to the opposite side.) The 90 degree angle is called a right angle. Two right angles form a straight line. Using similarity, we can derive the following property for right triangles.

Theorem 3. *If and only if ABC is a right triangle with side lengths a, b, and c where c is the hypotenuse, we have that $a^2 + b^2 = c^2$*

This property is known as The Pythagorean Theorem. Because the shortest distance between any two points is a straight line, The Pythagorean Theorem is great at finding the distance between points. If you can find a right triangle with a hypotenuse of some line segment, and if you know what the legs of the triangle are, then you can find the length of the line segment.

Example: Walker is 5 miles east and 12 miles north of MLG Swagville. He wishes to join his fellow Major League Gamers. What is the minimum number of miles he must walk to get to his destination?

Example: In the diagram, AB=13 cm, AD=5 cm, and DC=20 cm. What is the square of the length of AC? (Source: Introduction to Geometry AoPS)



It is also worth mentioning that the Pythagorean Theorem is an if and only if statement. This means that if a triangle has sides a, b, and c which satisfy $a^2 + b^2 = c^2$, then the triangle must be a right triangle.

Example: Point D is drawn on segment BC of triangle ABC such that $BD = 5, CD = 9, AB = 13$, and $AC = 15$. Find the length of segment AD.

3 Area of a Triangle

The area of triangles can be found in several ways depending on what information is given. Here are a few ways of calculating the area of a triangle.

3.1 Basic Method

Theorem 4. $A = \frac{bh}{2}$

This is the most prevalent method for finding the area of a triangle. Given a side-length and its altitude, the area is half their product.

3.2 Heron's Formula

Theorem 5. $A = \sqrt{s(s-a)(s-b)(s-c)}$, *where* $s = \frac{a+b+c}{2}$

This formula finds the area of a triangle given the side-lengths a, b, and c.

> **Example:** Oshadha is reading a book with triangular paper titled *The Four-Sided Triangle*. The book has 26 triangular pages, where each page is one sheet of triangular paper. The sides of the papers measure 52 inches, 41 inches, and 15 inches. What is the total area of paper for all 26 pages?

3.3 Trigonometric Method

Theorem 6. $A = \frac{1}{2}ab\sin C$

This formula finds the area of a triangle given the side-lengths a, b, and the angle between sides A and B, $\angle C$.

> **Example:** A triangle has sides of lengths 6 and 8, separated by an angle of 30°. What is the area of this triangle?

3.4 Equilateral Triangles

Theorem 7. $A = \frac{s^2\sqrt{3}}{4}$

This formula finds the area of an equilateral triangle with side-length s.

4 Special Triangles

Some types of triangles show up quite often in competitive problems. Often times, auxiliary lines - ones you add to the diagram - can make these special triangles, simplifying the problem solving process greatly. It is quite helpful to be able to recognize these special triangles when they come up and apply their properties.

4.1 45-45-90 Triangles

Isosceles right triangles hold an interesting property regarding their side lengths. Suppose that we have $\triangle ABC$, where $\angle B$ is a right angle. If we know the length of one of the legs (AB or BC) to have length s, we know that the other leg has length s (since it is an isosceles triangle). From that, we can use the Pythagorean Theorem to find the hypotenuse, $s\sqrt{2}$. This is true for all 45-45-90 triangles.

4.2 30-60-90 Triangles

Suppose that we have $\triangle ABC$, where $\angle B$ is a right angle and $\angle C$ is 30°. This triangle's special property also involves the side lengths of a triangle. If the side AB opposite of the 30° angle is of length s, then we can say $BC = 2\sqrt{3}$ and $AC = 2s$.

4.3 15-75-90 Triangles

This special triangle is a bit stranger than the rest. Suppose we have a $\triangle ABC$, where $\angle B$ is a right angle and $\angle C$ is 75°. If we were to draw an altitude from B to the hypotenuse AC to some point D, we would find an interesting property. If we know the hypotenuse AC to be of length s, we can say that $BD = \frac{s}{4}$.

4.4 Equilateral Triangles

The majority of the special properties of an equilateral triangle come from its own definition. Given one side length, we immediately know the rest. We can also find the area given only one side length, as we discussed previously. Another useful property of an equilateral triangle, however, is that it can be split into two 30-60-90 triangles.

5 Angle Bisector Theorem

The Angle Bisector Theorem states that, given $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{AB}{BD} = \frac{AC}{CD}$. Likewise, the converse of this theorem holds as well.

> **Example:** Walker owns a farm. On his farm, he has a sheep pen. The pen, denoted as ABC, is triangular and must be divided into an area for white sheep and an area for black sheep. For a point D on BC, the divider AD bisects angle A. If fence BC is 3 meters, fence AB is 2 meters, and the area set for the black sheep is twice as large as that set for the white sheep, what is length of fence AC?

6 Triangle Inequalities

Inequalities are extremely useful when it comes to dealing with variables without a specified value. They can be used with triangles as well, showing certain behaviors that show uniqueness or impossibilities within the figures.

6.1 Triangle Inequality

Theorem 8. *The Triangle Inequality states that in a triangle with side lengths a, b, and c, in no specific order, the sum of a and b must be greater than c. That is, $a + b < c$.*

> **Example:** If a triangle with an integral perimeter has side lengths 31, 42, and a, how many possible values are there for a?

6.2 Pythagorean Inequality

Theorem 9. *The Pythagorean Inequality states that in a triangle with side lengths a, b, and c, where $a < b < c$, we can determine one of the three following conditions:*

(1) If $a^2 + b^2 = c^2$, the triangle is right.
(2) If $a^2 + b^2 < c^2$, the triangle is obtuse.
(3) If $a^2 + b^2 > c^2$, the triangle is acute.

> **Example:** Given the side lengths, determine whether the following triangles are right, obtuse, or acute.
>
> 1. 20, 21, 29
> 2. 5, 12, 16
> 3. 6, 9, 12
> 4. 11, 60, 61
> 5. 14, 15, 16

7 Problems

1. Triangle ABC is an isosceles right triangle with the measure of angle A equal to 90 degrees. The length of segment AC is 6 cm. What is the area of triangle ABC, in square centimeters?

2. (1995 AHSME #23) The sides of a triangle have lengths 11,15, and k, where k is an integer. For how many values of k is the triangle obtuse?

3. Let ABC be a triangle with angle bisector AD with D on line segment BC. If $BD = 2$, $CD = 5$, and $AB + AC = 10$, find AB and AC.

4. In $\triangle ABC$, let P be a point on BC and let $AB = 20$, $AC = 10$, $BP = \frac{20\sqrt{3}}{3}$, and $CP = \frac{10\sqrt{3}}{3}$. Find the value of $m\angle BAP - m\angle CAP$.

5. In triangle ABC, $AB = 10$ and $AC = 17$. Let D be the foot of the perpendicular from A to BC. If $BD : CD = 2 : 5$, then find AD.

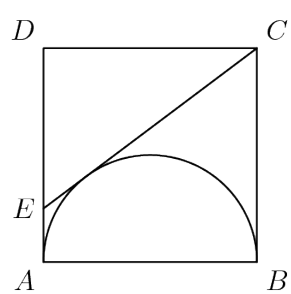
6. (2002 AMC 12B #20) Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

7. (2007 AMC 12A #6) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD?

8. Let $\triangle DEF$ be a right triangle with a right angle at $\angle E$ and $m\angle F = 75°$. If $DF = 20$, what is the area of $\triangle DEF$?

9. (2005 AMC 10B #14) Equilateral $\triangle ABC$ has side length 2, M is the midpoint of AC, and C is the midpoint of BD. What is the area of $\triangle CDM$?

10. Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to the semicircle from C intersects side AD at E. What is the length of CE?



11. The sum of the perimeters of two equilateral triangles is 45 inches, and the area of the larger one is 16 times the area of the smaller one. What is the area, in square inches, of the larger triangle?

12. (2006 AMC 10B #10) In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

13. (2009 AMC 10A #10) Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of $\triangle ABC$?

14. (2009 AMC 10B #20) Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets BC at D. What is BD?

15. (2005 AMC 12B #6) In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

16. We have a triangle $\triangle ABC$ such that $AB = 6$, $BC = 8$, and $CA = 10$. If AD is an angle bisector such that D is on BC, then find the value of AD^2.

17. Triangle ABC has three different integer side lengths. Side AC is the longest side and side AB is the shortest side. If the perimeter of ABC is 384 units, what is the greatest possible difference $AC - AB$?

18. (2006 AIME I # 1) In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal AC is perpendicular to CD, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

19. (2011 Mock Geometry AIME #6) Three points A, B, and C are chosen at random on a circle. The probability that there exists a point P inside an equilateral triangle $A_1B_1C_1$ such that $PA_1 = BC$, $PB_1 = AC$, and $PC_1 = AB$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

20. (2011 Mock Geometry AIME #13) In acute triangle ABC, ℓ is the bisector of $\angle BAC$. M is the midpoint of BC. a line through M parallel to ℓ meets AC, AB at E, F, respectively. Given that $AE = 1, EF = \sqrt{3}, AB = 21$, the sum of all possible values of BC can be expressed as $\sqrt{a} + \sqrt{b}$, where a, b are positive integers. What is $a + b$?