Sprint Test
Round 11100 - Solutions

1. D	6. B	11. E (1)	16. A	21. B	26. D
2. A	7. E (4024)	12. A	17. D	22. B	27. D
3. C	8. C	13. B	18. E $\left(\frac{100x}{100-x}\right)$	23. D	28. C
4. A	9. B	14. D	19. C	24. C	29. B
5. A	10. D	15. E (125)	20. C	25. C	30. C

5. The value is 1 for $x = 1$ or $x = 2010$. It is negative for all integers x in between.

9. Plug in $x = 2$ and $x = 3$ to get 2 equations with the 2 unknowns of $f(2)$ and $f(3)$. Solving this system of equations yields $f(2) = 7$.

12. Solving, the region is bounded by $y = \pm\frac{2}{3}x \pm 4$, and so it is a rhombus with vertices $(0, \pm 4)$, $(\pm 6, 0)$. It's area is therefore $\frac{12\cdot 8}{2} = 48$.

14. Letting $y = x^2 - 5x$ we have $0 = y^2 - 2y - 24 = (y-6)(y+4) = (x^2-5x-6)(x^2-5x+4) = (x-6)(x+1)(x-4)(x-1)$, so $x = 6, 4, \pm 1$, and the sum is 10.

15. From the given information we have $5 = b^2$ and $n = b^5$. Thus $bn = b^6 = (b^2)^3 = 5^3 = 125$.

17. $\log(17^{10000}) = 10000\log(17) = 12304.4...$, so there are 12305 digits.

23. Let x,y, and z be the dimensions. Then $x + y + z = 13$ and $2(xy + xz + yz) = 48$. Thus $\sqrt{x^2 + y^2 + z^2} = \sqrt{13^2 - 48} = 11$.

25. Let Q be the point of intersection of OP and RS. Let $OQ = a$ and $RQ = b$. Then $OR = \sqrt{a^2 + b^2}$. By similar triangles, $PR = \frac{b\sqrt{a^2+b^2}}{a}$. So $\frac{1}{PR^2} + \frac{1}{OR^2} = \frac{a^2}{b^2(a^2+b^2)} + \frac{1}{a^2+b^2} = \frac{a^2+b^2}{b^2(a^2+b^2)} = \frac{1}{b^2} = \frac{1}{16}$, so $b = 4$ and $RS = 2b = 8$.

26. The total area is 48. Using lines AC and AD to partition the figure, we have areas $ABC = 18$, $ADEF = 18$, and so $ACD = 12$. We need points G on BC and H on DE such that areas ACG and ADH are both 2. Since the heights of these triangles are 6 and 2, their bases must be $\frac{2}{3}$ and 2, respectively. So we have points $G(\frac{16}{3}, 6)$ and $H(8, 2)$. The sum of the slopes is therefore $\frac{9}{8} + \frac{1}{4} = \frac{11}{8}$.

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27. The function $f(x, y) = (x - y)^2$ does not satisfy property 3. For instance, $(6-0)^2 + (0 - (-3))^2 < (6-(-3))^2$. The rest of the functions indeed are "distance-like". The first two properties are easy to check. Property 3 holds for A by case analysis, for B because it is a scaling of the usual distance, and for C using the fact that $d(x, y) = |x - y|$ is "distance-like", and that if $a, b, c \geq 0$ with $a \leq b + c$ then it can be shown that $\frac{a}{a+1} \leq \frac{b}{b+1} + \frac{c}{c+1}$. (Multiply through by $(a+1)(b+1)(c+1)$ and subtract the left side to obtain $0 < abc + 2bc + b + c - a$, because $b + c > a$.)

28. Suppose M has k digits other than the leading digits 15. Then $M = 15 \cdot 10^k + r$ where r has k digits. Moreover, $5M = 100r + 15$ so simplifying we get $15 \cdot 10^k = 19r + 3$, or $15 \cdot 10^k \equiv 3 \mod 19$. Checking a few numbers, this implies $10^k \equiv 4 \mod 19$. Iterating multiplication by 10, the first power we find is $k = 16$, and hence M has 18 digits. [Alternatively, we can multiply 15 by 5 to discover the next digit (7), and then multiply 157 by 5 to discover the next digit, and so on until we find 15 again. See if you can show why this works, and what M is.]

29. If $x^4 + 2012x^2 + m = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a+c)x^3 + (ac+b+d)x^2 + (ad+bc)x + bd$, then we have $a + c = ad + bc = 0$, so $-a = c$ and $b = d$, and $2b - a^2 = 2012$, and $b^2 = m$. To minimize m we must minimize b, and since $a \neq 0$, b is minimized with $a = \pm 2$, so $b = 1008$ and $m = 1008^2$, which ends in a 4.

30. Choose a coordinate system so that L is the z-axis and P is the point $(3\sqrt{3}, 0, 0)$. Let $(p, q, r) \in X$. The distance from a point (p, q, r) to L is $\sqrt{x^2 + y^2}$ while its distance to P is $\sqrt{(p - 3\sqrt{3})^2 + q^2 + r^2}$. The given condition on the comparative distance from P to (p, q, r) and P to L is $p^2 + q^2 \geq 4((p - 3\sqrt{3})^2 + q^2 + r^2)$, which simplifies to $36 \geq 3\left(p - 4\sqrt{3}\right)^2 + 3q^2 + 4r^2$. Now translate by -$4\sqrt{3}$ in the x-direction. This does not change the volume. Letting $p' = p - 4\sqrt{3}$ we have the inequality $1 \geq \left(\frac{p'}{2\sqrt{3}}\right)^2 + \left(\frac{q}{2\sqrt{3}}\right)^2 + \left(\frac{r}{3}\right)^2$. This a solid ellipsoid centered at the origin, with a volume of $\frac{4}{3}\pi r_1 r_2 r_3 = 48\pi$.